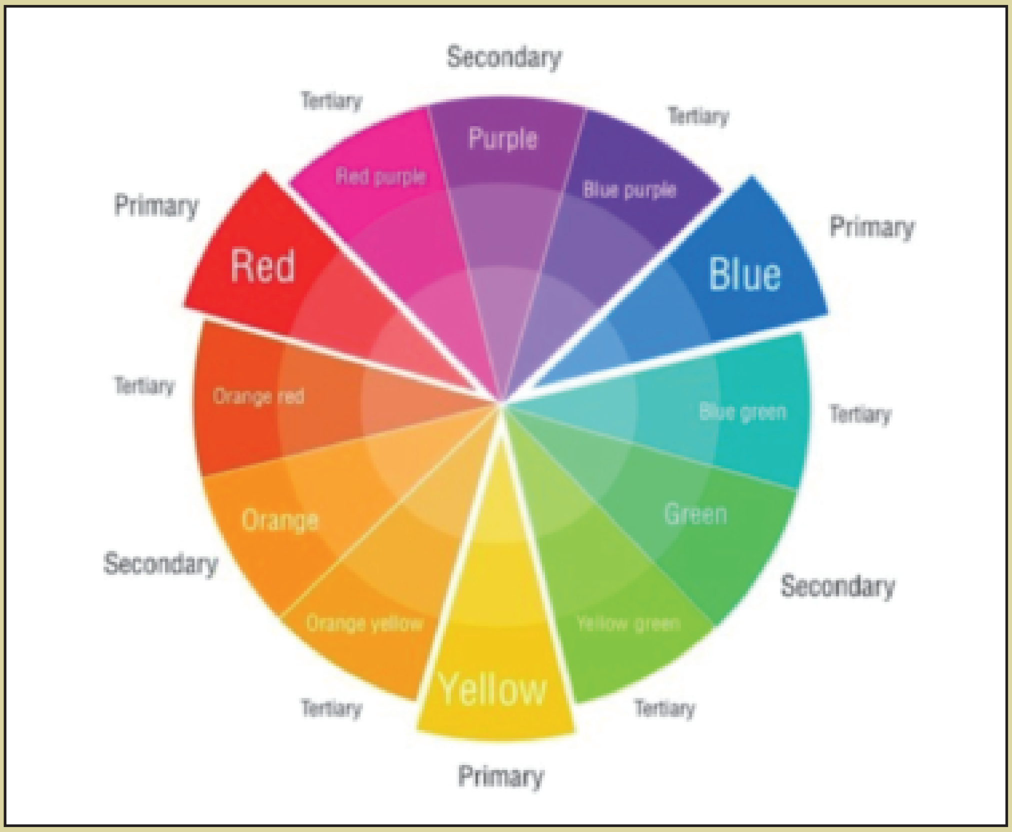
New York State

Mathematics Teachers' Journal



Mathematics in Color

Features:

Developing a Math Culture

On-line Peer Tutoring

Generalizing $(a+b)^2 = a^2+2ab+b^2$

And more…

VOLUME 67 Number 1 2017

New York State

Mathematics Teachers' Journal

Volume 67 Number 1 2017

Features

On the cover: See how to relate colors to mathematics in **Creating the Color Wheel** by Amy Seybold and Samantha Nickerson on p. 12 on this issue. Image found at http://www.brandisworld.com/images/amazing-paint-color-combinations-3-basic-color-wheel-600-x-450.jpg

An official Publicaton of
The Association of Mathematics Teachers of New York State
(An affiliate of the National Council of Teachers of Mathematics)

Journal Editor: **Robert Rogers, Ph.D.**
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MANUSCRIPTS submitted for publication should be addressed to the Editor. Manuscripts should be word processed using double spaced, 11 pt. Times New Roman font and should not exceed 14 pages in length. Figures, tables, and equations should be included in the text approximately where they should appear. Figures and tables should typically be labeled Figure 1, Table 2, etc., unless it is clearly referenced from the text. Since manuscripts are subject to a blind review, any author identification should appear only on a separate cover page. Three hard copies of the manuscript can be submitted to the Editor, or a WORD document may be attached to an email addressed to the Editor. Review of any submissions is at the sole discretion of the Editor.

PROBLEMS/SOLUTIONS:

Material for the Problems and Solutions section should be submitted directly to the Problems Editor.

MEMBERSHIP and SUBSCRIPTION:

Membership in AMTNYS is open to all individuals and institutions interested in the advancement of mathematics education from kindergarten through the university level.

ANNUAL DUES:

Individual/Institution:	$40
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The annual dues include a year's subscription to the *Journal*. Remittances should be made payable to *Treasurer, AMTNYS* and sent to the Data Manager. CHANGE OF ADDRESS should be reported immediately to the Data Manager.

AMTNYS Executive Board

From the Editor

Robert Rogers
Editor, NYSMTJ

The opinions expressed in this column are strictly those of the editor and do not necessarily reflect the views of the Association of Mathematics Teachers of New York State or its executive board.

After a ten-year hiatus from teaching freshman level calculus at the college level (not by choice), I started teaching University Calculus I and II in the fall of 2015 and spring of 2016. I have continued to teach these this academic year. This has fit into my own plans, as I have been collaborating with Eugene (Bud) Boman from Penn State - Harrisburg on a calculus book which "flips" the calculus course (not the class) by teaching the tools and the applications before going into the theory of limits, etc. This is actually more historically accurate, but this editorial is not about calculus nor directly about history. In fact, almost all mathematics exists because someone was trying to solve some specific problem. I've been allowing this premise to guide my teaching for some time, but this episode of teaching calculus has helped to galvanize this notion.

This semester, our department invited Heather Lewis and Yousuf George from Nazareth College to speak about their experiences teaching using Inquiry Based Learning (IBL). I was very impressed with their efforts to let their students drive the class by asking them the proper questions and letting the students run with it. Since I have been thinking about problem solving, in general, and calculus, specifically, I took the opportunity to share some of the materials Bud and I have developed, mainly problems developed. Heather responded by saying that we were *solving problems using calculus, not solving calculus problems*.

This really resonated with me, and as I said, galvanized my thoughts. It has really made me focus on using mathematics to solve problems rather than solving mathematics problems. This idea has transcended to all of my teaching, not just calculus, and I think it makes sense when teaching mathematics at all levels. We need to give students a reason to learn mathematics to solve problems of interest to them, rather than teach them mathematics techniques which are not necessarily of interest to them (or truthfully to most people). The analogy that I've used deals with my table saw. I could wax poetically about the virtues of my table saw, but unless you are a table saw fanatic, you probably would tune me out. But if you had a piece of wood you wanted cut, then maybe you might be interested in the attributes of my saw. The problems we use to "hook" our students need not be real life problems nor profound, but they must capture students' interest.

For example, all of my calculus students knew the colors of the rainbow: red, orange, yellow, green, blue, indigo, violet (ROY G BIV). I asked them offhandedly, "Which color is on the top of the rainbow, red or violet?" We looked on the internet to see that the red is on top. I then offhandedly asked if anyone had seen a double rainbow. Do you know that the colors in the secondary rainbow are reversed, and that the secondary rainbow is always above the primary rainbow? As soon as a student asked why this happens, I handed out a (carefully crafted) homework assignment which looked to explain these questions. This assignment utilized geometry, very little physics, and optimization. Again, it was a problem using mathematics, not a mathematics problem. It covered the main topic I wanted to get across, optimization, but it was on something the students could relate to. Students must still practice calculational exercises (which they do with an online homework system), but if we don't have them utilize these tools to solve problems of interest (which should be the initial motivation) then what is the point? You are back to my table saw. The analogy I have used for this last point is baseball. If one wants to play baseball, then (s)he must practice throwing, catching, batting, etc. However, it is hard to motivate someone to practice catching 100 grounders if (s)he has never seen a baseball game, nor ever has a chance to "get into the game." In fact, I tell my students that problems such as the rainbow problem are "the game." One must learn the mechanics of calculus, but this should not be construed as "the game." The game involves many tools, not just the techniques of calculus.

I've been talking about calculus, but I've been utilizing this idea in all my classes. For example, this semester I'm teaching a general education course. One of the topics to be covered is an introduction to matrices. Rather than just define a matrix and go through the rules for adding, multiplying, etc., I decided to look at the current discussion on health care. The question I asked them is, "Should health care be governed by the free market system or should the government intervene, and if so, then to what extent?" I made it clear to them that this issue is a complex one and we could not hope to give a definitive answer, but could we use mathematics to provide some insight into the issue? I won't go into the details on this (unless there is a request to write a future article on this), but it provides a natural way to examine graphing lines and utilizing matrices. In fact, the definition of matrix multiplication pops right out and, surprisingly, looks natural. This is a topic that could be done in a high school class, though it is not clear that high school students would be motivated by the topic of health care. The point is to present a problem first and then develop the mathematics it takes to do this problem. I've said in this column before that I've determined that my job as a teacher consists of three major components: motivation, support, and feedback. For me, the motivation part is crucial, as I must devise problems which I feel are interesting to students and will be pertinent to the topic I want them to learn. In ways, this is the most challenging part of my teaching and I give a lot of thought to this aspect. Again, the problem does not need to be real life. It could be as simple as noticing a pattern and utilizing mathematics to examine the pattern.

I don't want to sound flippant on this. Teachers cannot just examine topics as they wish. They have a set of topics that must be covered. The mechanics must be practiced to produce automaticity. Perhaps this is where collaboration with colleagues comes in. We need to be sharing our ideas, good or not so good, with others to see if they can adapt or adopt them. There is an internet out there with all types of materials/ideas on it. We are lifelong learners ourselves. Many of the things I'm using in class, I've learned within the last ten years. The rainbow problem was just adapted from a book within the past year. This is part of the challenge of teaching and is very exciting and satisfying. Trying to motivate learners is difficult, but is very rewarding. To me, there is no greater joy in my job than seeing a student understand mathematics through some problem that sparked his/her interest. Typically, I grow was well during the process. Common sense says that an interested, motivated student will learn better. We need to motivate students to want to learn mathematics to solve problems which interest them, not just learn the mathematics we want them to learn. As you probably already know, motivated students can do some amazing things.

Notes

- The Association of Teachers of Mathematics of NYC will be holding its 2017 Fall Conference on Saturday, November 18, 2017 at Hunter College. See www.atmnuc.org/homeconferences.html for details.

- The Mathematical Association of America Mathfest will be held on July 26-29, 2017 in Chicago, IL. See http://www.maa.org/meetings/mathfest-2017 for details.

- There will be a joint meeting of the Mathematical Association of America Seaway Section and the New York State Mathematical Association of Two Year Colleges on October 20-21, 2017at SUNY Broome. See https://people.rit.edu/maacway/meetings.html for details.

- The American Mathematical Association of Two Year Colleges will be holding its Annual Conference on November 9-12, 2017 in San Diego, CA. See http://www.amatyc.org/ for details.

See http://www.nctm.org/regionals/ for details,

The President's Message

David Hurst
President, Association of Mathematics Teachers of New York State

As I write this message, March Madness is winding down. No, I am not referring to the Men's and Women's NCAA basketball tournaments, but rather the AMTNYS affiliate conferences that have recently concluded. Close to two-thousand mathematics educators made their way to the AMTRA, TCMEA, LIMAÇON, and HMVAMA conferences this past March. Countless sessions sent participants on their way excited to inspire their students using new found ideas upon their return to the classroom. Educators also spent valuable time at the conferences developing their personal learning networks by informally sharing best practices with colleagues. For me, this is the power of AMTNYS; the building of a mathematics community.

As this school year winds down, continue to think of ways to inspire your students, make a difference in your school, or make a difference within the AMTNYS mathematics community. Your students are eager to learn. Inspire and enlighten them by making mathematics relevant and authentic. Be vocal about the great things you are doing in the classroom and share them with your colleagues. For it is through sharing that we can all implement positive changes that have the greatest possible impact on our students.

I encourage you to make time this summer to attend the second annual New^3 Summer Conference, "**Coming Together for Learning, Teaching, and Students**", July 9-12, 2017, at Siena College. This is a marvelous time to make connections with colleagues from not only around New York, but also from New Jersey and New England as well. If you cannot make the summer conference, then please consider attending our 67th Annual Fall Conference "**Full STEAM Ahead!**", November 3-4, 2017 at the Adams Mark in Buffalo.

Finally, as I have asked before, "How can you make a difference? What will be your legacy?" Please actively pursue the opportunities within AMTNYS to collaborate and grow. I encourage you to figure out how you can ramp up your participation. Consider volunteering on a conference committee and/or presenting at the Annual Fall Conference or Summer Conference. Please check www.amtnys.org often for information about becoming actively involved.

Summary of the 66th Annual AMTNYS Conference November 10-12 2016, Rye Brook NY

Ellen Falk
Conference Coordinator

"Road to the Core"

The 2016 AMTNYS Fall Conference was a great success and attended by more than 600 mathematics teachers from around the state. Responses from our post conference survey were extremely positive. People felt strongly that the conference provided opportunities for networking, collaboration and useful professional development. Based on feedback, we will also review our process for awarding the CTLE credits. The process was new to us and will hopefully be streamlined for the future.

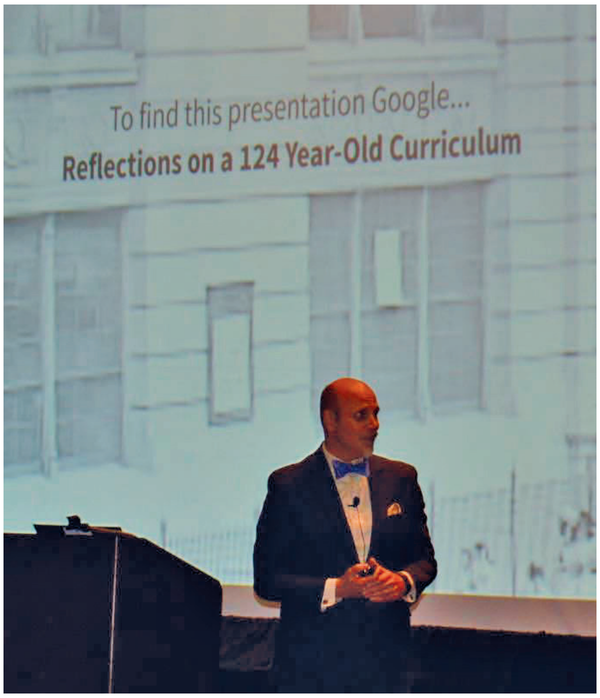
There are many people who contribute to the success of a conference and I am grateful to have had their guidance. The committee chairs and others did an outstanding job at pulling the conference together and I wish to thank: Donna Yerdon, Mike Siuta, Elizabeth Waite, Caryl Lorandini, Ronni David, Kate Martin-Bridge, Laurie Rosborough, Danielle Bouton-Wales, Bill Wales, Jim Matthews, Joe Straight, Jane Cushman, Susan Morse, Dana Morse, Keary Howard, Theresa Bartoy, Maria Michelsson, Stephanie Grasek and Elizabeth O'Donell. The keynote speakers, Dan Willingham and Nils Ahbel, provided inspiration and food for thought. John Svendsen and Mary Cahill represented NYSED and we appreciate their knowledge and involvement year after year.

There are two people behind the scenes I need to thank as well: Mike Gerver for the cover artwork and Mario Partenope for 10 projectors!

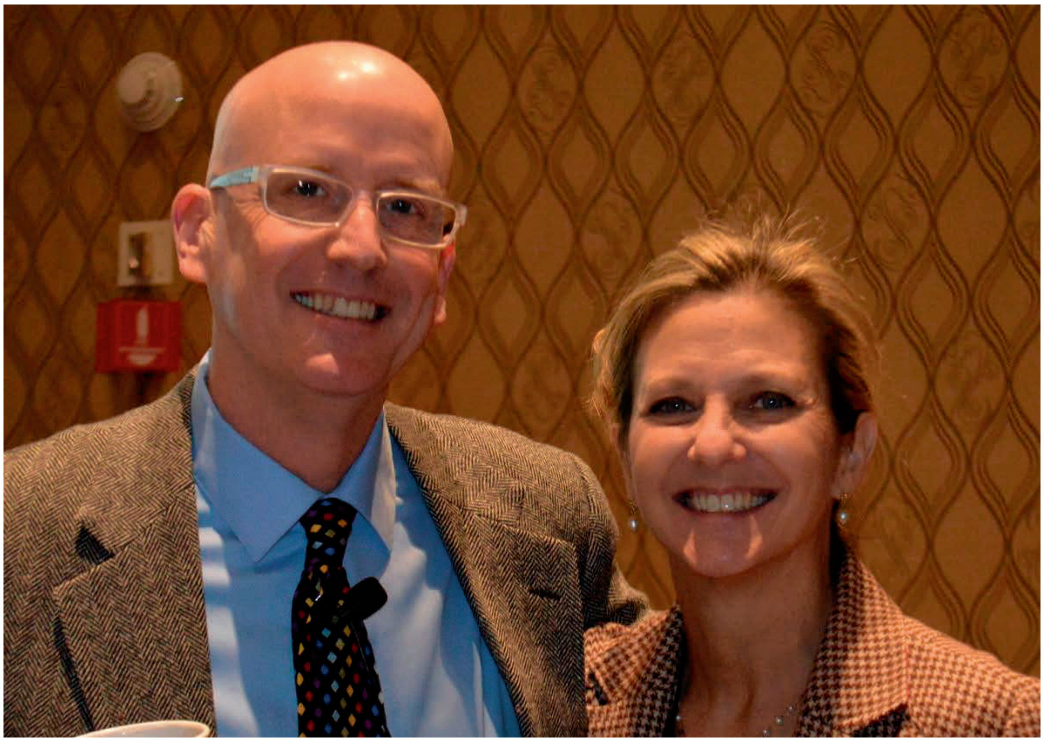
But most of all I wish to thank our AMTNYS members for attending and making our conference a success! Without you there wouldn't be a conference!

Scenes from the Conference

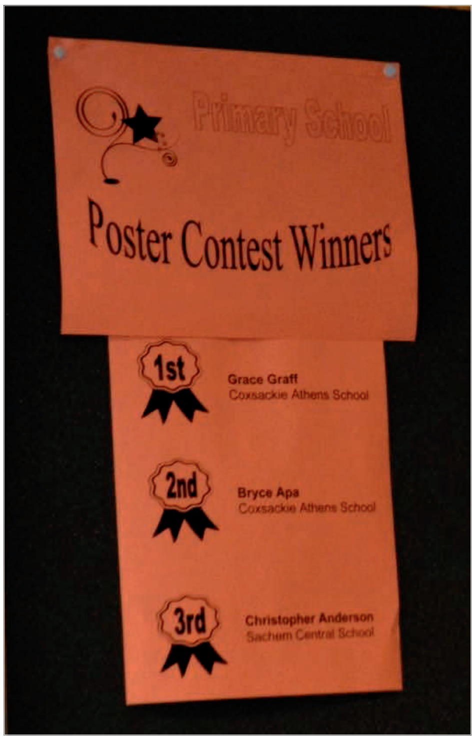
Immediate Past President Elizabeth Waite passes the gavel to incoming president David Hurst.

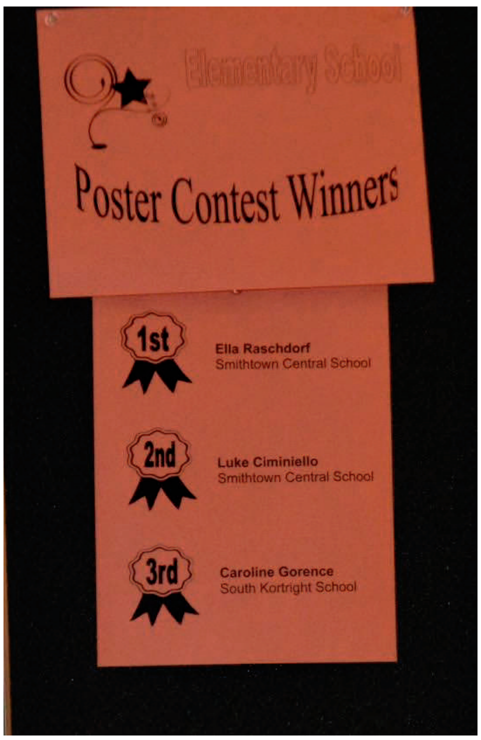


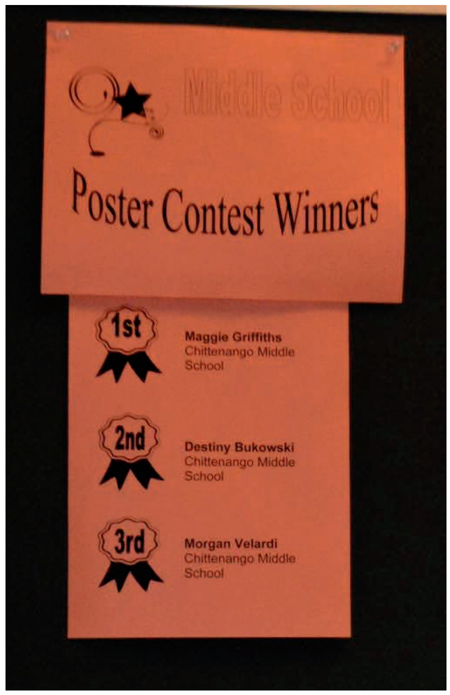
Nils Ahbel speaking at the banquet



Keynote speaker Dan Willingham with Conference Coordinator Ellen Falk







Poster Winners: Primary, Elementary, and Middle School

2016 AMTNYS Distinguished Service Award Winner – Frederick Roberts

Jim Matthews

Chair, Past President's Advisory Committee

The AMTNYS Distinguished Service Award is given annually to an AMTNYS member in recognition of extensive high quality service to the Association. The recipient of the award is selected by the Past Presidents' Advisory Committee from nominations submitted by AMTNYS members. AMTNYS Past Presidents, the current President and the President-elect are not eligible to receive the award. Previous awardees are: Gladys Hamilton, Ona Masters, Marie Muller, Marilyn Hanlon, Jean Dowd, Dick Gilbert, Fred Paul, Ernie Kelly, Helene Silverman, Elyne Schulte, Mary Genier, Rose Ambosino, Ardyce Elmore, Joe Manuel, Lynn Richbart, John Balzano, Sue Cloen, Linda Pearles, Richard Swanson, Mary Behr Altieri, Sheila Dolgowich, Sheila Bamberger, Laurie Rosborough, Beryl Szwed, Margery Masters, Tim Frawley, Susan McKenna, Frank Sobierajski, John Bailey, and Caryl Lorandini.

Jim Matthews with 2016 DSA winner Fred Roberts

Fred Roberts is the recipient of the 2016 AMTNYS Distinguished Service Award. Fred was selected from a group of outstanding nominees. The award, our Association's most prestigious honor, was presented to him by at the Association's 2016 Annual Meeting in Rye. Fred's wife, Donna, also a long-time contributor to AMTNYS and mathematics education was in attendance. Fred is the 31st person to be recognized with this award.

Fred earned his Bachelor's and Master's degrees in Secondary Mathematics Education at SUNY-Oswego. He also completed a Certificate in Educational Administration there. He began his teaching career in the Phoenix Central School District in 1972. At John C. Birdlebough High School, he taught everything from general mathematics to AP calculus and computer programming. After more than 23 years of teaching, he served for seven years as the Director of Secondary Curriculum, Technology, and Special Projects for the district. After retiring from Phoenix, Fred taught at Cayuga Community College for about ten years. In addition to his classroom teaching, Fred, in collaboration with his wife Donna, developed educational web pages including MathBits, Algegra2Bits, JuniorMathBits, and GeometryBits, all resources for educators.

Fred has presented and co-presented workshops at numerous AMTNYS and affiliate conferences for many years. Many of these sessions have been on using technology effectively in the classroom. Graphing calculators and computer software have been the focus of many of these sessions. Other presentations have focused on the Common Core. Perhaps the most popular are the sessions on Math and the Movies he leads with Donna.

For 25 years, Fred has served with distinction on the AMTNYS Professional Services Committee and for approximately half of this time, he served as chair of the committee. This committee is responsible for every aspect of the AMTNYS Scholarship program, including advertising the scholarship to over 60 colleges and universities in New York, collecting the applications, facilitating the selection process, hosting the scholarship winners at the annual meeting, and submitting an article to the journal about the honorees. Fred's service in this capacity has been exemplary.

For his many years of outstanding service to AMTNYS, it was with deep appreciation that our association honored Fred Roberts with the 2016 AMTNYS Distinguished Service Award.

2016 AMTNYS Scholarship Winners

Joan Koral
Chair, Professional Services Committee

Each year the scholarship committee requests applicants from college professors across New York State. From the bright, talented pool of future mathematics teachers, the four strongest candidates were selected to each receive an AMTNYS Scholarship of $2,000. The winners, presented below, were announced at the Fall Conference in Rye. Congratulations to our outstanding recipients!

The first scholarship winner, from Rochester NY, is **Allison Larter**, representing Houghton College. Allison will graduate this December with a degree in secondary mathematics education. As a resident assistant and active member of the college mathematics community at Houghton, she hopes to teach middle school mathematics. Dr. Yates, one of Allison's professors, championed her accomplishments in research, saying "I believe this experience shows that Allison will persist in the face of difficulties she encounters in the classroom as a teacher and will seek new and creative ways to help students learn the content of her courses."

The second scholarship recipient is **Ashley Thomas** representing SUNY, The College at Brockport and her hometown of Horseheads NY. Making the Deans list with honors, Ashley has balanced a full course load with extra curricular activities and a work-study job. "She has a clear vision for her future" says Pierangela Veneziani, Professor at SUNY, The College at Brockport. Ashley will graduate next December with a degree in secondary mathematics education.

The third winner, recipient of the Past President's AMTNYS Scholarship, is **Jessica Riesel** from Whitesboro NY, who attends Hartwick College. Jessica graduated with a degree in secondary mathematics education and special education this past May. Jessica's strong interest is in differentiated instruction and reaching students of varying abilities and skills. Dr. Davies, a professor at Hartwick College says of Jessica: "Her strengths lie in her incredible work ethic [and her] ability to examine and critically analyze logical arguments." Jessica hopes to teach middle school in the Oneonta area.

The fourth scholarship winner, selected as the Ona Masters Scholarship recipient, is **Zachary Norris** from Savannah NY, representing Roberts Wesleyan. A member of the men's Basketball team, Zachary hopes to motivate his students with enthusiasm and passion. He will graduate in May with a degree in secondary mathematics education. Jim Fisher, who worked with Zachary at a summer program, was impressed with his "boundless energy and creativity". Additionally, he adds "He had a joy about him that was infectious and brought the students and even staff along with him."

Appreciation & gratitude to the AMTNYS Executive Committee for their dedication to maintaining these awards and the Scholarship Fund Committee for their time and energy raising funds at various AMTNYS events throughout the year.

Many thanks are also extended to the Selection Committee, who dedicates time during the summer to make the difficult decisions of choosing our winners. An additional thank you to the Past Presidents' Group, without whose support there would be one less scholarship.

A final thank you to AMTNYS members who support the awards through donations and the purchase of AMTNYS merchandise. If you would like more information on the AMTNYS Scholarship Program, please visit the web site http://www.amtnys.org/index.html.

STEM: Connects Learning

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Problems and Solutions for the Elementary Grades

Jamar Pickreign
Associate Editor

In this edition, we offer three new problems for your consideration. Feel free to use these either in your class or as part of a math circle or math club. Please send your students' solutions, problem suggestions or questions you have: regular mail (or email) to: Jamar Pickreign, 200A Sibley Hall, 101 Broad Street, SUNY Plattsburgh, Plattsburgh, NY, 12901 (jamar.pickreign@plattsburgh.edu). We will accept solutions to these new problems submitted any time before October 15, 2017.

As before, Teachers are responsible for supplying parental permission for the publication of individual student names, or, alternatively, as "students in Mrs./Ms./Mr./Dr. Teachers Name's class offer the following solutions." Handwritten solutions and accompanying photographs/diagrams are acceptable for submissions.

Don't forget, anyone may work on these problems, but only solutions from students K-6 will be recognized. Partial and complete solutions will be acknowledged with the most "appealing" solutions published. Emphasize to your students that partial solutions in solving problems are a valuable part of the problem solving process and we encourage their submission, though we expect grade appropriate explanations for solutions to be considered complete. Have fun and direct any questions to Jamar Pickreign at the above. We are also accepting suggestions for new problems for consideration.

PROBLEMS

Elementary Problem 32: Unit Fractions

A unit fraction is a fraction whose numerator is 1. Examples are $\frac{1}{2}, \frac{1}{3}, \frac{1}{12}$, etc. Notice that $\frac{3}{4} = \frac{1}{2} + \frac{1}{4}$.

In how many ways can each of the following fractions be written as a sum of exactly two *different* unit fractions?

$$\frac{1}{2} \quad \frac{2}{3} \quad \frac{3}{4} \quad \frac{5}{6} \quad \frac{6}{7} \quad \frac{7}{8}$$

Elementary Problem 33: Eight Sticks

You have eight sticks of the following lengths $1, 2, 3, 4, 5, 6, 7, 8$. How many different triangles can be made by choosing 3 of these sticks?

Elementary Problem 34: Birthday Puzzle

Sally and Joe were born on the same day, but in different years. This year, Sally has six times as many candles on her birthday cake than Joe. In 12 years, Sally will have twice as many candles on her cake than Joe will have. How many candles do Sally and Joe have on their cakes this year?

Make It - Take It

One of the popular activities at the annual AMTNYS conferences is the Make It - Take It session. In this session, pre-service teachers provide classroom ready take-aways for conference attendees. This series will attempt to recreate this experience as well as can be accomplished in a print form. After a brief introduction, the actual materials that were distributed at the conference will be included. Feel free to adopt or adapt these activities/lessons. Also, feel free to share your own classroom ready materials. You can send your ideas to Rebecca Conti, Department of Mathematical Sciences, SUNY Fredonia, Fredonia, NY 14063 or submit them electronically to rebecca.conti@fredonia.edu.

Creating the Color Wheel

Amy Seybold Samantha Nickerson

SUNY Fredonia

Introduction

Proportional reasoning is a concept of mathematcs that has been found to be one of the more difficult tasks for some students. This concept also plays a huge role in other concepts moving forward in mathematics, so it may be helpful to look at the many ways of reading proportions such as fractions and percentages. Ratios are a way to express the proportions of components in a combination. Combining these two concepts together allows for students to see a variety of representations between the numbers they are comparing. The goal of this task is to provide a good context for engaging students in reasoning about proportions and ratios. This interactive exercise focuses on using what you know about ratios and equivalent ratios to make a prediction about whether the student can understand and work out the paint recipes for each of their colors.

References

"Standards for Mathematical Practice." Home | Common Core State Standards Initiative. Web. 21 Sept. 2016.

Beswick, K., "Make Your Own Paint Chart", http://files.eric.ed.gov/fulltext/EJ921981.pdf

Rainbow Ratios by Meg Lamm and Susan Jennings

Creating the Color Wheel

Grade Level: K-2

Materials:

- Circular paper
- Primary color paint - Red, Yellow, Blue
- Q-tips (or paintbrushes)
- Pencil
- Pallet

Common Core State Standard:

Understand and apply properties of operations and the relationship between addition and subtraction.

CCSS.MATH.CONTENT.1.OA.B.3
Apply properties of operations as strategies to add and subtract.2 Examples: If 8 + 3 = 11 is known, then 3 + 8 = 11 is also known.

Learning Objectives:

1. Students will be able to identify the three primary colors with 100% accuracy by painting each in their color wheel.
2. Students will be able to reproduce a secondary color with 100% accuracy by mixing the same amount of two primary colors on their color wheel.
3. Students will be able to predict what two primary colors combined make with 90% accuracy by showing the commutative property of addition within colors.

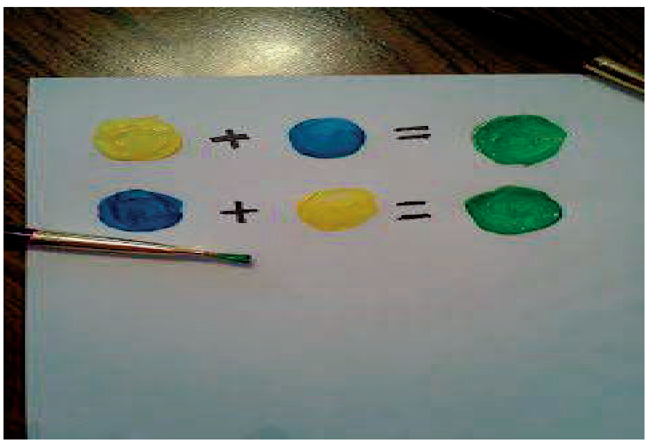
How it works:

The teacher will distribute color wheels (small circular paper) to each of the students. The teacher will direct students to fold their color wheel in half once, in half again, and in half a third time, in order to make 6 different sections on their color wheel. Students will each get a pallet of paint, including the colors red, yellow, and blue - the primary colors, as well as three q-tips to mix each of the colors. Since the q-tips are two-sided, students can use one side for each of the primary colors and the other side to mix the paint when they create the secondary colors - purple, orange, and green. The color wheel will be set up so that all three primary colors have an empty space in between, to make the mixing process easier and clearer to recognize.

After the students have painted their initial three sections of primary colors, they will begin mixing two of the primary colors to create a secondary color in the triangle that comes between the two. Students will mix yellow and red to create orange, yellow and blue to create green, and blue and red to create purple. This process will continue until the entire triangle is shaded in with its respective color.

Extension:

To extend this activity, students will look at the concept of the Commutative Property of Addition. To look at this activity more in depth, we are essentially adding things together. For example, $Yellow + Blue = Green$ and $Red + Yellow = Orange$. The Commutative Property of Addition states that the addends can be added in any order and the sum will still be the same. We will want to show that $Yellow + Blue = Green$ will give the same result as $Blue + Yellow = Green$. Students will begin by showing their two addends on one side of the equal sign but with their order switched. Students will then test out the theory by mixing the paints in the order they have written their equation out to see if they still produce the same answer.



Fraction Wheel

Grade Level: 3-5

Common Core State Standard:

Develop understanding of fractions as numbers.

CCSS.MATH.CONTENT.3.NF.A.3

Explain equivalence of fractions in special cases, and compare fractions by reasoning about their size.

CCSS.MATH.CONTENT.3.NF.A.3.A

Understand two fractions as equivalent (equal) if they are the same size, or the same point on a number line.

CCSS.MATH.CONTENT.3.NF.A.3.B

Recognize and generate simple equivalent fractions, e.g., 1/2 = 2/4, 4/6 = 2/3. Explain why the fractions are equivalent, e.g., by using a visual fraction model.

Learning Objectives:

1. Students will be able to identify fractions and their decompositions with 100% accuracy by cutting out the pieces from their fraction wheel.
2. Students will be able to reason about fraction sizes with 90% accuracy by producing the equivalent or nonequivalent fractions with their respective fraction wheel.
3. Students will be able to predict which value of a fraction is greater than or less then with 90% accuracy by using fraction wheel to compare and contrast fraction value.

Materials:

- Paper plates
- Markers
- Paint
- Paint brushes
- Scissors

How it works:

As students continue to work with primary and secondary colors, they will further their knowledge as they begin to demonstrate fractions. Using each of the six colors they have started with and created – red, yellow, and blue, purple, orange, and green – the students will start to develop fraction sense as they paint each of their paper plates with a specific color. So that all students are doing the same thing, each fraction will be designated to a color. Paper plates will serve as the wheel, with one kept as whole, and one divided into halves, thirds, fourths, fifths, and sixths. We will designate the colors:

$$Red \rightarrow 1\ whole$$
$$Blue \rightarrow \frac{1}{2}$$
$$Yellow \rightarrow \frac{1}{3}$$
$$Green \rightarrow \frac{1}{4}$$
$$Orange \rightarrow \frac{1}{5}$$
$$Purple \rightarrow \frac{1}{6}$$

Students will first cut their plates into the correct number of fractions we will be going to make. Since we are working with six colors of paint, we will be working with only six plates. They will demonstrate that each of their six places are cut in their respective piece number, e.g., 1 whole, 2 halves, 3 thirds, 4 fourths, 5 fifths, and 6 sixths. With their paint and paintbrushes, students will begin to paint their fraction wheel with the color that was designated for each fraction. Students can then begin to look at the fraction as a whole as well as the number of pieces it is composed of.

Extension:

After students have a grasp on the concept of the simplified fractions, they can begin to make equivalent fractions by multiplying the numerator and the denominator by the same value. For example, if we are working with $\frac{3}{5}\,orange$, we can then take that $\frac{3}{5}$ and multiply it by $\frac{2}{2}$. To show that, we will have $\frac{3}{5} \cdot \frac{2}{2} = \frac{6}{10}$ and therefore we will have the equivalent fraction of $\frac{3}{5} = \frac{6}{10}$, which students will then begin to learn about at the sixth and seventh grade level dealing with ratios and proportions.

Painting with Proportions

Because some days, it is okay to get a little messy in the classroom!

Grade Level: 6-8

Common Core State Standard:

Understand ratio concepts and use ratio reasoning to solve problems.

CCSS.MATH.CONTENT.6.RP.A.2

Understand the concept of a unit rate associated with a ratio with, and use rate language in the context of a ratio relationship.

Materials:

- Cups
- Red, yellow, and blue tempera paint
- Popsicle sticks
- Teaspoons, or some type of measurement for the paint
- Plastic aprons (optional)
- Pre-mixed paint cups
- Paint brushes
- Paper
- Table cloth or newspaper

Learning Objective:

Students will be able to convert their ratio of colors into an equivalent fraction with 100% accuracy by writing down a correct proportion.

How it works:

Each student will get a cup with a color of paint that was already mixed up by the teacher ahead of time from the three primary colors; red, blue, and yellow. Students will have to mix their own paint to try to match the pre-made paint color by determining the ratio of each primary color in the combination. They will be given a tool to measure the ratios, such as a teaspoon or dropper. Once the student figures out the ratios of the primary colors in the "recipe" of the color they were given, the student must give the color that they mixed up the most creative name that they can think of! They can, then, paint the name that they came up with and their ratio in reduced form. The teacher can also have the students write their ratio in different forms, such as $\frac{a}{b}$, $a\colon b$, and a to b.

Example:

Maybe the student was given a container of purple paint. The student will need to determine what colors were used to make up the purple color by mixing colors and matching it, and they might find that the ratio of the red paint to the blue paint that matches is $6\colon 3$. Thus, on their paper they might name your paint "Purplicious Grape", and write the ratio in simplest form $2\colon 1, \frac{2}{1}$ and 2 to 1.

Developing a Culture of Mathematics at the High School Level (and All Levels)

Jillian Folino
Indian River High School

Editor's Note: Jillian describes the development of a mathematical culture at the high school level, but the ideas transcend grade level. Many of the activities can be adapted to any grade level and even the mathematics competitions have middle school and elementary school counterparts. For example, the MAA American Mathematics Competition has one at the eighth grade level [http://www.maa.org/math-competitions/amc-8] and there is the Math Olympiads for Elementary and Middle School (MOEMS) [http://www.moems.org/]. A culture of learning and enjoying mathematics can be fostered at all levels.

How did Math Club become one of the biggest clubs at Indian River High School? I've often found myself and others asking the same question. It is a widespread belief that if you offer food at a high school club meeting, students will sense it from miles away and file into your classroom eager to find out what's on the "agenda." We stopped offering food at regular club meetings two years ago, and yet they still come. At the beginning of this school year, I found myself at a math scavenger hunt event for Math Club that I had somehow forgotten to advertise. I hadn't put out a single announcement or poster on the wall, however, I looked around at the twenty high school students eagerly going from classroom to classroom looking for an irrational number and was again puzzled with why they decided to come. After two full years of Math Club events, going on the third, I didn't realize the answer until a colleague in the mathematics field finally explained it to me, we had developed a culture of mathematics at Indian River. That's why they come!

MATH CLUB SCAVENGER HUNT

Directions: Your goal is to find all of the items on the list below in the fastest amount of time. You must take a selfie with the items so that each member of your group is visible. Your photographs MUST go in order (that doesn't mean you need to take them in order, you can rearrange them after).

Only ONE photo can be taken from the inside pages of a textbook and NO photos can be taken OF a computer or electronic device. You cannot print pages and take a photo of the pages printed or write anything down and take a photo of it.

You must return to room 222 by 3:05. Late entries will not be accepted. All photos must be submitted by 3:10. The team with the most correct photos will win. In the event of a tie, the team who returned the fastest will win.

Categories:

1) Irrational Number
2) Negative Integer
3) Whole Number
4) Fraction
5) Decimal
6) Square Root
7) Number line
8) Number bigger than 999
9) Number smaller than -20
10) Number between 9.9 and 9.8
11) A Greek letter
12) A rhombus
13) A pentagon
14) An obtuse angle
15) Two parallel lines
16) Vertical Angles
17) An equilateral triangle

In the fall of 2014, Indian River's Math Club officially started as a partnership with SUNY Potsdam and with the financial support of the Mathematical Association of America's Dolciani Mathematics Enrichment Grant. From the beginning, we wanted the club to have a non-curriculum based mathematical focus that incorporated fun activities that were accessible for all grade levels and abilities. We started the club with an event rather than a meeting to bring publicity and encourage participation. An announcement about a Math Club meeting being held after school doesn't sound nearly as enticing as a "team math jeopardy competition for all grade levels with food and prizes!" While formal meetings took place throughout the year, the focus was more on simple and inexpensive mathematics based events that were open to the entire student population. Our events ranged from snowflake making and origami to the mathematics of bubbles. We added a "best math costume" to the already existing Halloween costume contest, had mathematical scavenger hunts throughout the building, and had regular brain teaser nights where students could work on physical or mental puzzles. While these events were all open to the general student population and often popular, we also saw the number of regular members of Math Club grow rapidly.

If you asked the students, they would say there are three main highlights throughout the year for math club. The first is our participation in the American Mathematics Competition at SUNY Potsdam. The students take a field trip to the college where they compete, but also get an opportunity to meet other math minded high school students, explore a college campus, and spend some time thinking about their future beyond high school.

MATH CONTESTS

- American Mathematics Competitions (formerly American High School Math Exam) is the first stage of contests for selecting the US National Team for the International Math Olympics
- 75 minute multiple choice problem solving challenges
- Two levels – AMC10 for grades 9-10 and AMC12 for grades 11-12
- Offer it in your high school or find a local college where you can take students to compete (www.maa.org/math-competitions). Visiting a college has other benefits too!

The second is Pi Week and all the events surrounding that one irrational week. With the help of the students, Math Club decorated the hallways, hosted a pie the teacher competition, and cheered on students in a pie eating competition just to start! The students also sponsored a competition with the teachers to see who could best incorporate mathematics into other disciplines! The biggest highlight for me as an advisor was the daily math question posted on the announcements, in which we had hundreds of students participate daily. At the end of the week I had a social studies teacher say to me in passing "My first period students don't even want to start class until they've discussed the math question of the day and begged me to let them go to your room to submit an answer. I wish I could get my students this excited about history!"

PI WEEK DAILY TRIVIA QUESTIONS

Monday: When written out in letters, what is the largest number between 0 and 100 that does not contain the letter *e*?

Tuesday: There are several books on a bookshelf. If the math book is the 3rd from the left and the 8th from the right, how many books are on the shelf?

Wednesday: What is the angle between the minute hand and the hour hand at 3:30?

Thursday: A baseball and a bat cost $110 together. If the bat costs $100 more than the ball, how much does the ball cost?

Friday: Tom says that he can give you any exact amount of change between 1 cent and 1 dollar with the coins in his pocket. If tom has the least amount of coins possible, how many pennies, nickels, dimes and quarters does he have in his pocket? (Note: half dollar coins are not being used)

The biggest highlight for the students is their reward at the end of the year where we take students to Darien Lake for Math/Physics Day at the Amusement Park! The students work on some mathematics on the way to and from the park and engage with other math minded students from across the state. It is a day of fun, math and bonding that really helps students feel like a valuable part of Math Club.

Indian River's Math Club is going strong on its 3rd year with more than thirty active members and much more participation throughout the school population during its now traditional school wide events! Students are more engaged in mathematics than they were four years ago, and are provided an opportunity to see the world of mathematics beyond the lens of the academic classes they take. I see students that rarely complete my homework in class eagerly participating in mathematics as an extra-curricular event. I have watched current Math Club members bringing their younger friends and siblings to events and telling them how much fun we have in Math Club! These comments and actions from students are what really tell me that we have achieved our goal of getting students to appreciate and enjoy mathematics. My students joke about my enthusiasm inside the classroom towards mathematics and ask me frequently if I sit at home at night doing math problems for fun. Of course, I always laugh and humor them, but what they don't realize is they are doing just that. They might not see it that way and, yes, it's still at school and not at home on their couch, but the absolute best moments are when students see the mathematics they are doing as fun, and don't see it as academic at all.

TIPS FOR STARTING A MATH CLUB

- Focus on fun!
- Highlight math outside of the regular curriculum
- Exploit their interests
    - Food & prizes
    - Short events at convenient times
    - Social interactions
- Make a splash! Start with a large scale fun events!
- Have activities that can reach a broad audience

At the start of the next school year, I know I will have a handful of kids that I haven't met yet asking when Math Club is starting. I will have students that miss a meeting and come to me without any prompting to ask what they missed and when the next event is. What started as a personal goal when I thought I was going to be running a club has become a reality for me as an actual advisor for this truly student run club. Not all the kids are destined to have a career in a mathematics field or even take calculus in high school, but I do know that all of my kids appreciate mathematics more, and are more likely to explore mathematics beyond high school because of their experience in Math Club.

RESOURCES

- New York State Mathematics Teacher' Journal
- National Council for Teachers of Mathematics Journal
- Mathematical Association of America – American Mathematics Competitions

Continued from p. 28

3. Mathworld- A Wolfram Resource, Wolfram Research Inc, Champaign-Urbana, IL, (2015)

4. The OEIS (On-Line Encyclopedia of Integer Sequences), The OEIS Foundation, ATT Research, Murray Hill, NJ (2015)

5. Ribenboim, Paulo, *The New Book of Prime Number Records, 3rd Edition,* Springer-Verlag (1995)

6. Rosen, Kenneth, *Elementary Number Theory and its Applications, 5th Edition,* Pearson/Addison Wesley (2005)

Gimball - A Math Peer-Tutoring Social Network for e-Community Service

Jonathan Collard de Beaufort
William Alexander Middle School, Brooklyn, NY

Editor's Note: Jonathan's Father contacted me about Jonathan's efforts to create an e-community support service for fellow students and invited me to check it out. I signed up to gain access and was impressed with this eighth grader's initiative, enthusiasm, and technical acumen. I encouraged Jonathan to write about his experiences creating this page. You might want to check it out to see if you and your students can make use of it. Of course, no teaching technique is universal, but it is good to have as many tools in your teaching toolbox as possible. This adds a new wrinkle to the tried and true method of students helping students which you and your students may find appealing.

As an 8th-grade student who loves math, I see every day how my classmates can struggle in that subject. For example, some are too shy to ask questions during class or during sessions teachers make available to them, so they go on without properly understanding the material. Very few use private tutors; private tutors are very expensive, especially here in New York City.

Many of my friends would email or call or text or Instagram or Snapchat me in the evening or early in the morning at times. I have often had to answer the same questions, and I have felt that I need to juggle with too many different messaging tools. But the point is that many students ask classmates for help.

Getting help from other students has several benefits. For one, students probably feel less judged. Also, they can phrase their questions in an informal way and without necessarily using the terminology that a teacher would expect. The student providing the help has an incentive, too; typically, these tutoring sessions count toward their community service requirements.

For all these reasons, and more, peer tutoring is arguably one of the more effective complements to learning in class. A study by the Education Endowment Foundation ranks peer tutoring as the third most efficient among 10 education strategies considered in the study [The Economist, 2016].

Finding tutoring students for each student in need is not efficient, however. When tutor and tutee do meet, a lot of time is wasted in transportation or waiting for each other. Moreover, they can't meet very often, making it virtually impossible for struggling students to have fast help on questions big and small, let alone at the exact moment they do their homework. Finally, I would argue that the separation between struggling students and tutors is artificial: students at the same grade levels can be in either situation depending on their understanding of the current topic, and can be, in turn, tutor and tutored. This is what peer tutoring is all about.

A website venue to constantly match tutors and students needing help can solve most of these problems. Teaching videos and even now MOOCs (Massive Open Online Courses) are well known. However, they do not provide help tailored to each student's questions. Fortunately, a blog format allows every participant to post his/her questions or to answer someone else's. If the site has enough participants, then questions should receive timely answers. Also, sharing equations and formulas over the internet is cumbersome: HTML has tags for math formulas, but they are so tedious and time consuming that their use makes sense only if you expect many people to read your text, and if you have time. A website dedicated to math must solve that problem. Here, a tutor talks to a single person and will not waste his or her precious time on typing.

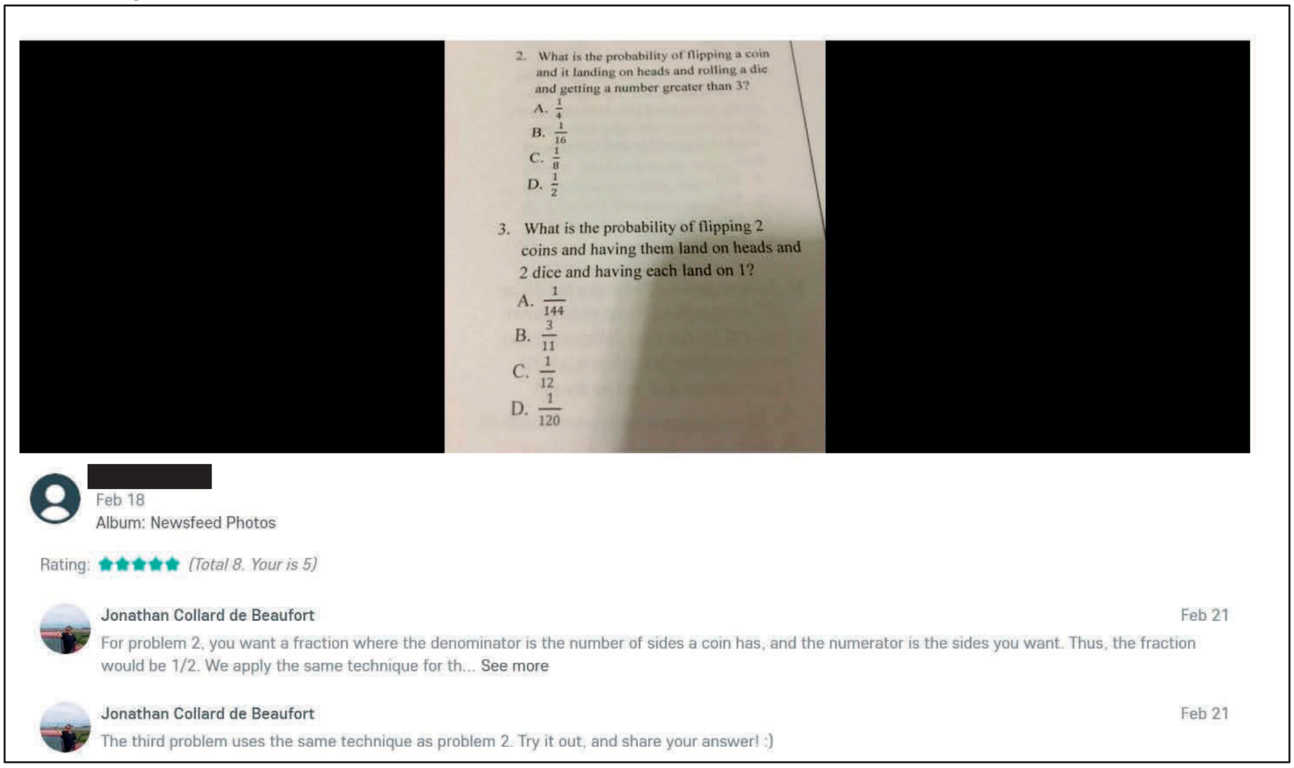
To solve these issues, I created a math peer-tutoring social network for e-community service called Gimball [www.gimball.org]. Math problems and equations can be posted by taking a picture of a textbook paragraph or a homework hand-out. A student can snap a picture with his/her cell phone and post it with a few words. In a few seconds, the problem is posted. Likewise, a student who can suggest an answer can write it down on a piece of paper and snap a sketch of his/her solution. Thus, users can expect to receive help on their questions early enough to finish their homework or to be ready for the next test.

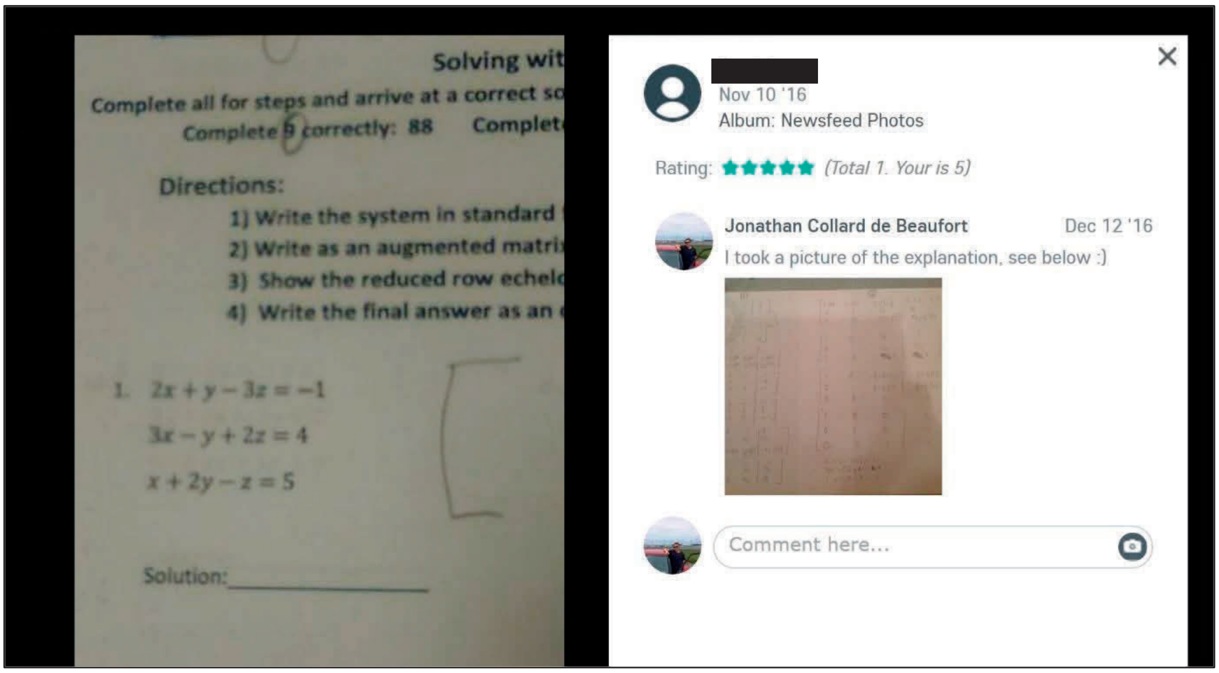
Also, since tutors want to get rewarded for their work, Gimball uses a system where the best and most active users get "starred." Students most helpful to their virtual math community will be distinguished and receive awards. If requested, we will inform their schools' principals and recommend that they get community service credit for the time spent e-tutoring.

I also wanted Gimball to be a safe site. Emails are verified and users are examined. In addition, posts are reviewed to make sure that there isn't any inappropriate content that isn't math related. Moreover, everything on Gimball is public. There is no private messaging, making sure that nothing is happening in the "shadows." Thus, Gimball is kid-safe.

Finally, Gimball is a useful tool for teachers. Teachers can create groups for their classes, and post challenges for their students to solve. With this tool, teachers can examine how well students know the topic, and see how well they articulate what they are thinking. Gimball isn't meant as a substitute to teachers or school, but as a complement to teaching. Gimball's usefulness however depends on the number of participants and their activity on it. As such, I encourage mathematics teachers to check out the site and ask their students to sign up if they think it will help. School principals should also be encouraged to give credit for e-community service on Gimball. Then, the benefits of Gimball will automatically snowball!

Following are some screenshots from Gimball; names other than the author's are blacked out.





Reference

Teaching the Teachers, *The Economist*, June 11, 2016, http://www.economist.com/news/briefing/21700385-great-teaching-has-long-been-seen-innate-skill-reformers-are-showing-best

Delightful Interconnections between Pythagorean Triples and Recursive Sequences

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Abstract

This article will serve to furnish interconnections between Pythagorean triples and recursive sequences including Fibonacci-like sequences, as well as explore some neat patterns embedded in the table of Pythagorean and Primitive Pythagorean triples.

Pythagorean Triple

A Pythagorean triple is a set of positive integers (x, y, z) satisfying the equation $x^2 + y^2 = z^2$. For example, the trio $(11, 60, 61)$ comprises a Pythagorean triple since $11^2 + 60^2 = 3721 = 61^2$. Moreover, a Pythagorean triple is designated primitive (heretofore abbreviated PPT) if the $\gcd(x, y, z) = 1$. The trio $(11, 60, 61)$ provides one with an example of a PPT. For the majority of applications, PPT's are far more interesting than all Pythagorean triples which arise from integer multiples of PPT's; for if (x, y, z) is any Pythagorean triple satisfying the relation $x^2 + y^2 = z^2$, then (kx, ky, kz) is likewise for any positive integer k. To see this, one merely observes that $(kx)^2 + (ky)^2 = k^2(x^2 + y^2) = k^2z^2 = (kz)^2$.

It is well known that all PPT's arise from the following set of generating equations:

$$x = m^2 - n^2, \quad y = 2mn\,, \quad z = m^2 + n^2$$

such that m and n obey the following conditions: m and n are positive integers of opposite parity in the sense that one of the integers is even and the other is odd, $m > n$ and $\gcd(m, n) = 1$. The standard proof that these equations are both necessary and sufficient to generate all primitive PPT's, while not difficult, is fairly lengthy and a proof can be found in the excellent number theory textbook by Kenneth Rosen [6]. In contrast, it should be noted that not all Pythagorean triples are generated by this formula. For example, consider the Pythagorean triple (15,20,25). This triple, obtained by multiplying each of the sides of the PPT (3,4,5) by 5, is missed by our generating formula. To see this, note that $20 = 2mn$ implies $10 = mn$. The only integer solutions are $m = 5, n = 2$ or $m = 10, n = 1$. We see in Table 1 that that neither of these generating values for m and n yields the desired Pythagorean triple (15,20,25). Let's further examine Table 1, which provides all the PPT's for $m \leq 12$. Prime numbers are highlighted in red.

m	n	$x = m^2 - n^2$	$y = 2mn$	$z = m^2 + n^2$	$PPT(x, y, z)$
2	1	3	4	5	$(3, 4, 5)$
3	2	5	12	13	$(5, 12, 13)$
4	1	15	8	17	$(15, 8, 17)$
4	3	7	24	25	$(7, 24, 25)$
5	2	21	20	29	$(21, 20, 29)$
5	4	9	40	41	$(9.40, 41)$
6	1	35	12	37	$(35, 12, 37)$
6	5	11	60	61	$(11, 60, 61)$
7	2	45	28	53	$(45, 28, 53)$
7	4	33	56	65	$(33, 56, 65)$
7	6	13	84	85	$(13, 84, 85)$
8	3	55	48	73	$(55, 48, 73)$
8	5	39	80	89	$(39, 80, 89)$
8	7	15	112	113	$(15, 112, 113)$
9	2	77	36	85	$(77, 36, 85)$
9	4	65	72	97	$(65, 72, 97)$
9	8	17	144	145	$(17, 144, 145)$

m	n	$x = m^2 - n^2$	$y = 2mn$	$z = m^2 + n^2$	$PPT(x, y, z)$
10	1	99	20	101	$(99, 20, 101)$
10	3	91	60	109	$(91, 60, 109)$
10	7	51	140	149	$(51, 140, 149)$
10	9	19	180	181	$(19, 180, 181)$
11	2	117	44	125	$(117, 44, 125)$
11	4	105	88	137	$(105, 88, 137)$
11	6	85	132	157	$(85, 132, 157)$
11	8	57	176	185	$(57, 176, 185)$
11	10	21	220	221	$(21, 220, 221)$
12	1	143	24	145	$(143, 24, 145)$
12	5	119	120	169	$(119, 120, 169)$
12	7	95	168	193	$(95, 168, 193)$
12	11	23	264	265	$(23, 264, 265)$

Table 1: A brief table of PPT's

Observe that $y = 2mn$ is divisible by 4 since m and n are of opposite parity. If m or n is divisible by 3, then so is y. Otherwise 3 is a factor of $x = m^2 - n^2$ (since the remainder of both m^2 and n^2 when divided by 3 would be 1). Noting that $\gcd(x, y) = 1$, exactly one of these components is divisible by 3. As a consequence, the product of the legs in any PPT is divisible by 12. Moreover, exactly one of the three components in any PPT is divisible by 5. This is because the remainder of any perfect square, when divided by 5, is $0, 1$, or 4. This will ensure that at least one of $2nm$, $m^2 - n^2$, or $m^2 + n^2$ is divisible by 5. The fact that $\gcd(x, y, z) = 1$ ensures that exactly one does. Consequently, the product of the three components is divisible by 60. Observe further that $z \equiv 1 \bmod 4$. To see this, recall that in a PPT, m and n are of opposite parity. Hence either m is even and n is odd or vice versa. In the first case, $m^2 \equiv 0 \pmod 4$ and $n^2 \equiv 1 \pmod 4$, so $z = m^2 + n^2 \equiv 1 \pmod 4$. In the second case, $m^2 \equiv 1 \pmod 4$ and $n^2 \equiv 0 \pmod 4$, so $z = m^2 + n^2 \equiv 1 \pmod 4$.

Recursive sequences enter the study

The Fibonacci sequence is one of the most fascinating sequences in mathematics for both its intrinsic beauty and wide range of applicability to numerous disciplines including architecture, art and the social sciences. Thousands of identities and a plethora of patterns are prevalent in the sequence recursively defined as follows: Let F_n denote the n^{th} term in the Fibonacci sequence where we define $F_1 = F_2 = 1$ and $F_n = F_{n-2} + F_{n-1}$ for $n \geq 3$. The first fifteen terms of the Fibonacci sequence are $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377$ and 610. The next goal is to illustrate the interconnection of this sequence to Pythagorean triples. Consider any four consecutive terms in the Fibonacci sequence. First form the product of the first and fourth terms selected. Next take twice the product of the second and third terms. Finally form the sum of the squares of the second and third terms. One always obtains a Pythagorean triple! To cite two examples, first consider the four consecutive terms $3, 5, 8$ and 13. Applying the steps, we obtain $3 \cdot 13 = 39$; $2 \cdot 5 \cdot 8 = 80$; $5^2 + 8^2 = 89$. This computation leads one to the PPT $(39, 80, 89)$ which is a Pythagorean triple since $39^2 + 80^2 = 7921 = 89^2$. As a second example, consider the four consecutive terms $8, 13, 21$ and 34. Observe that $8 \cdot 34 = 272$; $2 \cdot 13 \cdot 21 = 546$; $13^2 + 21^2 = 610$ and $272^2 + 546^2 = 372100 = 610^2$. Notice that the Pythagorean triple $(272, 546, 610)$ is not primitive since 2 is a common factor of all three components in the triple.

We next verify that this procedure always produces a Pythagorean triple. Let x and y be the first two terms anywhere in the Fibonacci sequence. By the recursion relation, the next two terms are necessarily $x + y$ and $y + (x + y) = x + 2y$. If we apply the aforementioned steps, we obtain

$$x \cdot (x + 2y) = x^2 + 2xy\,; \quad 2y \cdot (x + y) = 2xy + 2y^2\,; \quad y^2 + (x + y)^2 = x^2 + 2xy + 2y^2$$

Moreover,

$$\begin{aligned}(x^2 + 2xy)^2 + (2xy + 2y^2)^2 &= (x^4 + 4x^3y + 4x^2y^2) + (4x^2y^2 + 8xy^3 + 4y^4)\\ &= x^4 + 4x^3y + 8x^2y^2 + 8xy^3 + 4y^4\\ &= (x^2 + 2xy + 2y^2)^2\end{aligned}$$

Hence one has the Pythagorean triple $(x \cdot (x + 2y),\ 2y \cdot (x + y),\ y^2 + (x + y)^2)$.

Some observations

The pattern in the Fibonacci sequence is odd, odd, even, odd, odd, even, … with the even terms three places apart. Let $F_m, F_{m+1}, F_{m+2}, F_{m+3}$ be four consecutive Fibonacci numbers. If the first and last terms in this sequence of four consecutive Fibonacci numbers are odd, then the parity for the consecutive terms is odd, odd, even, odd or odd, even, odd, odd. We will show that in this case a PPT is formed. To see this, we will first streamline our notation by letting $\gcd(m,n)$ be denoted by (m,n). Using this notation, we will utilize the following fact about the greatest common divisor of Fibonacci numbers: $(F_m, F_n) = F_{(m,n)}$. See [1] for a proof of this. Applying this to our sequence of four consecutive Fibonacci numbers, we have $(F_m, F_{m+1}) = F_{(m,m+1)} = F_1 = 1$. Similarly, we have $(F_m, F_{m+2}) = F_{(m,m+2)}$. Notice that if d divides into m and $m+2$ then d divides into $(m+2) - m = 2$, thus $d = 1$ or $d = 2$. In either case, we have that $F_{(m,m+2)} = F_1 = F_2 = 1$. Thus F_m and F_{m+2} are relatively prime as well. With a similar argument, we have $(F_m, F_{m+3}) = F_{(m,m+3)} = F_1 = 1$ or $F_3 = 2$. If we assume that F_m and F_{m+3} are odd, then $(F_m, F_{m+3}) = 1$.

Applying these results to our previous four consecutive Fibonacci numbers $x, y, x+y, x+2y$, we have that these are all relatively prime. Thus, the two legs of our generated Pythagorean triple $x \cdot (x+2y)$ and $2y \cdot (x+y)$ are relatively prime. This is utilizing the fact x and $x+2y$ are both odd. It is straightforward to show that if any two terms of a Pythagorean triple are relatively prime, then all three are. (Any prime number which divides two of them will automatically divide the third.) Furthermore, notice that if x is even, then the legs of the Pythagorean triple are both even and the Pythagorean triple is not primitive.

It is important to realize that any Fibonacci-like sequence where the starting two terms are different, but the recursion rule is the same such as the Lucas sequence defined recursively by $L_1 = 1, L_2 = 3,\ L_n = L_{n-1} + L_{n-2}\ ,\ n \geq 3$ will produce Pythagorean triples using the same three steps. The first fifteen terms in the Lucas sequence are $1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843$ and 1364.

In Table 2, we furnish the first twelve Pythagorean triples produced by four consecutive terms in the Fibonacci sequence and the Lucas sequence. Observe that the generating values n and m are the second and third terms respectively among the four consecutive terms in the Fibonacci-like sequence $(m-n, n, m, m+n)$. The Fibonacci Pythagorean triples are highlighted in dark blue and the Lucas Pythagorean triples in green. PPT's produced are marked by $(+)$ immediately to the right of the Pythagorean triple.

m	n	$x = m^2 - n^2$	$y = 2mn$	$z = m^2 + n^2$	$PT\ (x,y,z)$
2	1	3	4	5	$(3,4,5)\ \ (+)$
3	2	5	12	13	$(5,12,13)\ \ (+)$
5	3	16	30	34	$(16,30,34)$
8	5	39	80	89	$(39,80,89)\ \ (+)$
13	8	105	208	233	$(105,208,233)\ \ (+)$
21	13	272	546	610	$(272,546,610)$
34	21	715	1428	1597	$(715,1428,1597)\ \ (+)$
55	34	1869	3740	4181	$(1869,3740,4181)\ \ (+)$
89	55	4896	9790	10946	$(4896,9790,10946)$
144	89	12815	25632	28657	$(12815,25632,28657)\ \ (+)$
233	144	33553	67104	75025	$(33553,67104,75025)\ \ (+)$
377	233	87840	175682	196418	$(87840,175682,196418)$
4	3	7	24	25	$(7,24,25)\ \ (+)$
7	4	33	56	65	$(33,56,65)\ \ (+)$
11	7	72	154	170	$(72,154,170)$
18	11	203	396	445	$(203,396,445)\ \ (+)$
29	18	517	1044	1165	$(517,1044,1165)\ \ (+)$

m	n	$x = m^2 - n^2$	$y = 2mn$	$z = m^2 + n^2$	$PT\ (x, y, z)$
47	29	1368	2726	3050	(1368, 2726, 3050)
76	47	3567	7144	7985	(3567, 7144, 7985) (+)
123	76	9353	18696	20905	(9353, 18696, 20905) (+)
199	123	24472	48954	54730	(24472, 48954, 54730)

Table 2: Pythagorean triples produced from four consecutive terms

Observe that the hypotenuses of all the Pythagorean triples formed when n and m are consecutive Fibonacci numbers belong to the Fibonacci sequence. This is not an accident. The Fibonacci identity $F_n^2 + F_{n+1}^2 = F_{2n+1}$ provides justification. A proof of this identity can be found in R. Honsberger's book [2] as well as in Mathworld [3]. This identity does not carry over to the Lucas sequence. We next consider a number of investigations and raise a number of thought provoking questions based on the findings in Tables 1 and 2.

Some interesting findings based on Tables 1 and 2

An analysis of Tables 1 and 2 leads one to consider the following questions:

1. When do the hypotenuse and the longer leg in a PPT differ by one?
2. When do the hypotenuse and one of the legs in a PPT differ by two?
3. When do the two legs in a PPT differ by one?
4. When is one of the legs in the PPT a prime number?
5. When are both of the odd components in a PPT prime numbers?

Before examining these questions, let us observe that the difference between the hypotenuse and the even leg in a PPT is the square of the difference between the generators m and n. Likewise, the difference between the hypotenuse and the odd leg in a PPT is twice the square of the smaller generator. The proofs of these follow since

$$z - y = m^2 + n^2 - 2mn = (m - n)^2$$
$$z - x = m^2 + n^2 - (m^2 - n^2) = 2n^2$$

At this juncture, we turn to resolve our questions.

To determine when the hypotenuse and the longer leg in a PPT differ by one, it suffices to solve the equation $z = y + 1$. In light of the previous paragraph, we have $(m - n)^2 = z - y = 1$, so $m - n = \pm 1$ or equivalently $m = n \pm 1$. Since $m > n$, then $m = n + 1$.

To determine when the hypotenuse and one of the legs in a PPT differ by two, notice that x and z are both odd, while y is even. Thus, it suffices to solve the equation $z = x + 2$ or equivalently $z - x = 2n^2 = 2$. Thus we have, $n = 1$ and thus $x = m^2 - 1$, $y = 2m$, $z = m^2 + 1$. Since x and z are both odd, then m must be even.

To determine when the two legs in a PPT differ by one, we need to solve the equations $y = x \pm 1$ or equivalently $2mn = m^2 - n^2 \pm 1$. Using the quadratic formula, we solve these two equations for m obtaining the respective solutions $m = n + \sqrt{2n^2 \pm 1}$. The values of m and n that lead to integer solutions are members of the Pell sequence recursively defined as follows: $P_1 = 1$, $P_2 = 2$, $P_n = 2 \cdot P_{n-1} + P_{n-2}$. The first twelve terms in the Pell sequence are $1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741$ and 13860.

In order for a leg to be a prime number, the leg must be odd so that $x = m^2 - n^2 = (m - n)(m + n)$. Thus $m - n = 1$ and $m + n$ is a prime number. Now $m - n = 1$ implies $m = n + 1$. The condition is necessary, but not sufficient; for if $m = 5$ and $n = 4$, then $x = m^2 - n^2 = 5^2 - 4^2 = 9$ which is not a prime number.

In order that the hypotenuse and the odd leg are prime numbers requires that both $x = m^2 - n^2$ and $z = m^2 + n^2$ are prime numbers. As Paulo Ribenboim's book on prime numbers indicates, it has been conjectured that there are infinitely many PPT's whose odd components x and z are both prime numbers [5].

To conclude, we furnish the first twelve occurrences of positive outcomes that illustrate the ideas for the five questions posed above.

m	n	$x = m^2 - n^2$	$y = 2mn$	$z = m^2 + n^2$	$PPT\ (x, y, z)$
2	1	3	4	5	$(3,4,5)$
3	2	5	12	13	$(5,12,13)$
4	3	7	24	25	$(7,24,25)$
5	4	9	40	41	$(9.40,41)$
6	5	11	60	61	$(11,60,61)$
7	6	13	84	85	$(13,84,85)$
8	7	15	112	113	$(15,112,113)$
9	8	17	144	145	$(17,144,145)$
10	9	19	180	181	$(19,180,181)$
11	10	21	220	221	$(21,220,221)$
12	11	23	264	265	$(23,264,265)$
13	12	25	312	313	$(25,312,313)$

Table 3: The hypotenuse and the longer leg differ by one

If one examines the table, (s)he observes that when m and n differ by one, the odd legs form an arithmetic progression with common difference two. This is no accident and can easily be verified algebraically. Note that $m = n + 1$ implies

$$[(m+1)^2 - m^2] - [m^2 - (m-1)^2] = [m^2 + 2m + 1 - m^2] - [m^2 - m^2 + 2m - 1] = 2$$

The closed formula generating the sequence of hypotenuses in this instance is given as follows:

$$z = m^2 + n^2 = m^2 + (m-1)^2 = 2m^2 - 2m + 1$$

m	n	$x = m^2 - n^2$	$y = 2mn$	$z = m^2 + n^2$	$PPT\ (x, y, z)$
2	1	3	4	5	$(3,4,5)$
4	1	15	8	17	$(15,8,17)$
6	1	35	12	37	$(35,12,37)$
8	1	63	16	65	$(63,16,65)$
10	1	99	20	101	$(99,20,101)$
12	1	143	24	145	$(143,24,145)$
14	1	195	28	197	$(195,28,197)$
16	1	255	32	257	$(255,32,257)$
18	1	323	36	325	$(323,36,325)$
20	1	399	40	401	$(399,40,401)$
22	1	483	44	485	$(483,44,485)$
24	1	575	48	577	$(575,48,577)$

Table 4: The hypotenuse and one of the legs in the PPT differ by two

If one examines Table 4, there are simple closed formulas for the three components x, y, z. Since m is even, we let $m = 2k$ for some integer k. Combining this with the fact that $n = 1$, we have

$$x = m^2 - n^2 = 4k^2 - 1\,, \qquad y = 2mn = 4k\,, \qquad z = m^2 + n^2 = 4k^2 + 1$$

m	n	$x = m^2 - n^2$	$y = 2mn$	$z = m^2 + n^2$	$PPT\ (x, y, z)$
2	1	3	4	5	$(3, 4, 5)$
5	2	21	20	29	$(21, 20, 29)$
12	5	119	120	169	$(119, 120, 169)$
29	12	697	696	985	$(697, 696, 985)$
70	29	4059	4060	5741	$(4059, 4060, 5741)$
169	70	23661	23660	33461	$(23661, 23660, 33461)$
408	169	137903	137904	195025	$(137903, 137904, 195025)$
985	408	803761	803760	1136689	$(803761, 803760, 1136689)$
2378	985	4684659	4684660	6625109	$(4684659, 4684660, 6625109)$
5741	2378	27304197	27304196	38613965	$(27304197, 27304196, 38613965)$
13860	5741	159140519	159140520	225058681	$(159140519, 159140520, 225058681)$
33461	13860	927538921	927538920	1311738121	$(927538921, 927538920, 1311738121)$

Table 5: The two legs differ by one

Observe that the hypotenuses of the right triangles are all Pell numbers. In general, $P_n^2 + P_{n+1}^2 = P_{2n+1}$. This theorem can be proven in a similar manner as the related Fibonacci Identity alluded to in Section 3. For example, $P_3^2 + P_4^2 = 5^2 + 12^2 = 169 = P_7 = P_{2\cdot 3+1}$. In other words, the sum of the squares of two consecutive Pell numbers coincides with the Pell number which is the sum of the subscripts of the two Pell numbers.

m	n	$x = m^2 - n^2$	$y = 2mn$	$z = m^2 + n^2$	$PPT\ (x, y, z)$
2	1	3	4	5	$(3, 4, 5)$
3	2	5	12	13	$(5, 12, 13)$
6	5	11	60	61	$(11, 60, 61)$
10	9	19	180	181	$(19, 180, 181)$
15	14	29	420	421	$(29, 420, 421)$
30	29	59	1740	1741	$(59, 1740, 1741)$
31	30	61	1860	1861	$(61, 1860, 1861)$
36	35	71	2520	2521	$(71, 2520, 2521)$
40	39	79	3120	3121	$(79, 3120, 3121)$
51	50	101	5100	5101	$(101, 5100, 5101)$
66	65	131	8580	8581	$(131, 8580, 8581)$
70	69	139	9660	9661	$(139, 9660, 9661)$

Table 6: Both components of odd parity are prime numbers

Next steps

This article served to incorporate primitive Pythagorean triples, number theory, and patterns entailing recursive sequences. One might be inclined to determine why no Pythagorean triple exists consisting of all three sides that are Fibonacci numbers, when the hypotenuse or legs in a right triangle are perfect squares, or the number of PPT's having a given hypotenuse. The reader is likewise invited to explore additional patterns and form new conjectures along these lines. With a plethora of websites and computer technology, new ground can be broken generating further palatable results.

References

1. Hoggatt, V.E., *Fibonacci and Lucas Numbers*, Dover Publishing (1969)

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Continued on p. 20

$(a+b)^2=a^2+2ab+b^2$: A Special Case

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One of the first identities that first-year students of algebra run into involves the expansion of the square of the sum of two variables. With a and b as the two variables, that identity looks like:

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad (1)$$

Eventually, most of those students get used to the term, $2ab$, in the expansion and the fact that, despite the identities $2(a+b) = 2a + 2b$ and $(ab)^2 = a^2b^2$, the operation of exponentiation is not distributive over the operation of addition.

The initial "proof" that equation (1) was an identity that I favored as a classroom teacher was the geometric approach. [See Figure 1.] In fact, the way I initially taught students to simplify expressions like $(2x+3)^2$ was to have them get the areas of the rectangles in a diagram like the one in Figure 2 and then add the like terms so they literally saw that $(2x+3)^2 = 4x^2 + 12x + 9$.

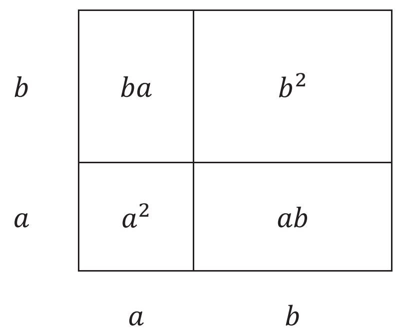


Figure 1

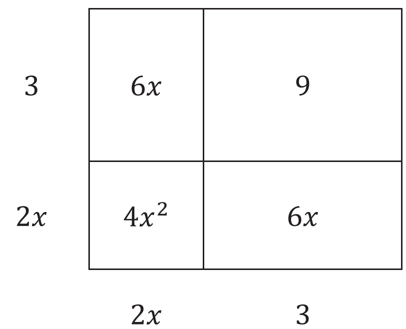


Figure 2

Recently, I had occasion to think about the identity (1) at the same time I ran across the well-known identity for perfect squares of positive integers

$$1 + 3 + 5 + \cdots + (2n-1) = n^2$$

A diagram of this identity is shown in Figure 3.

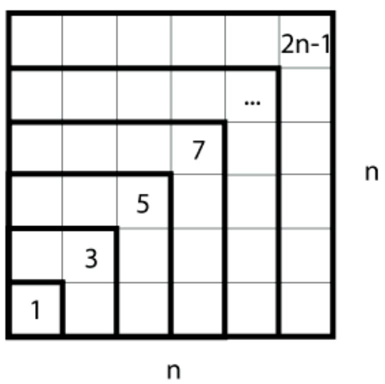


Figure 3: The sum of the first n odd integers http://www.9math.com/images/oddsum.png

This prompted me to recall the definition of triangular numbers, the sums of the first n positive integers. Using the notation $n^{(t)}$ for the n^{th} triangular number, we have

$$n^{(t)} = 1 + 2 + 3 + \cdots + n$$

[It should be noted that this is not the standard notation for the n^{th} triangular number, which is typically denote by T_n. It should also be noted that $n^{(t)}$ is not an exponential expression.]

This thought, in turn, made me wonder what the relationship would be between the values $(a+b)^{(t)}$ and $a^{(t)} + b^{(t)}$ when a and b are positive integers. Readers may want to explore this question on their own before moving on to my findings.

I started with a random selection of values for a and b and came up with the results in Table 1.

a	b	$(a+b)^{(t)}$	$a^{(t)}$	$b^{(t)}$	$a^{(t)} + b^{(t)}$	$(a+b)^{(t)} - \left(a^{(t)} + b^{(t)}\right)$
3	4	28	6	10	16	12
2	5	28	3	15	18	10
1	5	21	1	15	16	5
3	6	45	6	21	27	18
4	7	66	10	28	38	28

Table 1

It didn't take long to notice that the values for $(a+b)^{(t)} - \left(a^{(t)} + b^{(t)}\right)$ were the products of the corresponding values of a and b. This suggested that the equation

$$(a+b)^{(t)} = a^t + ab + b^{(t)} \qquad (2)$$

was also an identity. A simple algebraic argument using the familiar formula for triangular numbers, $n^{(t)} = \frac{n^2+n}{2}$, proved that such was the case.

$$(a+b)^{(t)} = \frac{(a+b)(a+b+1)}{2} = \frac{a^2+ba+ab+b^2+a+b}{2} = \frac{a(a+1)}{2} + \frac{2ab}{2} + \frac{b(b+1)}{2} = a^{(t)} + ab + b^{(t)}$$

The two sums, $1+2+\cdots+n$ and $1+3+\cdots+(2n-1)$, are both examples of arithmetic series; in the former case, both the first term and the common difference are 1; in the latter, the first term is also 1, but the common difference is 2. In general, the arithmetic series with first term 1 and common difference k,

$$1 + (1+k) + (1+2k) + \cdots + (1+(n-1)k)$$

has as its sum what is termed the n^{th} k-gonal number. When $k = 1, 2, 3$ and 4, those values are usually called the triangular, square, pentagonal and hexagonal numbers, respectively [Figure 4].

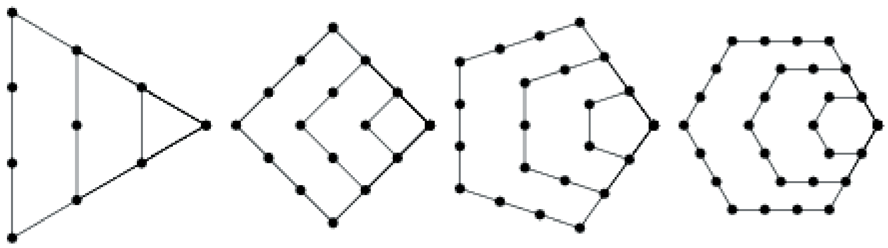


Figure 4: Triangular, square, pentagonal, and hexagonal numbers
http://mathworld.wolfram.com/images/eps-gif/PolygonalNumber_1000.gif

I wondered, would I come across identities similar to (1)and (2) if I worked with arithmetic series whose first terms were 1 and common differences were arbitrary positive integers k? For example, what identity might arise if I worked with the values of the arithmetic series whose first terms were 1 and common difference was 3: $1 = 1$, $5 = 1 + 4$, $12 = 1 + 4 + 7$, … . Before searching for such an identity, I generalized the notation I would use to look for an answer to such questions. Because the values $1, 1+k, 1+2k, \ldots$ are the values of an arithmetic progression, I decided upon

$$n^{(kp)} = 1 + (1+k) + (1+2k) + \cdots + (1+(n-1)k) \qquad (3)$$

Again, a warning that (kp) is not an exponent. Thus $n^{(1p)} = n^{(t)}, n^{(2p)} = 1 + 3 + \cdots + (2n-1) = n^2$, and $n^{(3p)}$ represents the sum $1 + (1+3) + (1+6) + \cdots + (1+(n-1)3)$. Using the same values for a and b that I used in Table 1, I came up with the values found in Table 2:

a	b	$(a+b)^{(3p)}$	$a^{(3p)}$	$b^{(3p)}$	$a^{(3p)} + b^{(3p)}$	$(a+b)^{(3p)} - \left(a^{(3p)} + b^{(3p)}\right)$
3	4	70	12	22	34	36
2	5	70	5	35	40	30
1	5	51	1	35	36	15
3	6	117	12	51	63	54
4	7	66	22	70	92	28

Table 2

Once again, it didn't take long to suspect that an identity similar to (1) and (2) applied to values of the form $n^{(3p)}$; namely,

$$(a+b)^{(3p)} = a^{(3p)} + 3ab + b^{(3p)}$$

This, along with what we have so far:

$$(a+b)^{(1p)} = (a+b)^{(t)} = a^{(t)} + ab + b^{(t)} = a^{(1p)} + 1ab + b^{(1p)}$$
$$(a+b)^{(2p)} = (a+b)^2 = a^2 + 2ab + b^2 = a^{(2p)} + 2ab + b^{(2p)}$$

made me hopeful that a more general identity existed for values of the form $n^{(kp)}$, when k was a positive integer; namely

$$(a+b)^{(kp)} = a^{(kp)} + kab + b^{(kp)} \quad (4)$$

A proof that my hope was warranted follows.

To start, note how $n^{(kp)}$, defined in equation (3), can also be expressed.

$$n^{(kp)} = 1 + (1+k) + (1+2k) + \cdots + (1+(n-1)k) = n + k\big(1 + 2 + \cdots + (n-1)\big)$$
$$= n + k\frac{(n-1)n}{2} = n + k\frac{(n^2-n)}{2}$$

Accordingly,

$$(a+b)^{(kp)} = a + b + k\frac{(a+b)^2 - (a+b)}{2}$$
$$= a + b + k\frac{a^2 - a + 2ab + b^2 - b}{2}$$
$$= \left(a + k\frac{a^2-a}{2}\right) + kab + \left(b + k\frac{b^2-b}{2}\right)$$
$$= a^{(kp)} + kab + b^{(kp)}$$

What about $(a-b)^2 = a^2 - 2ab + b^2$?

Of course, since it's also true that $(a-b)^2 = a^2 - 2ab + b^2$ is an identity, I initially thought that the equation $(a-b)^{(kp)} = a^{(kp)} - kab + b^{(kp)}$ would be another identity. The first case I tried, $a = 4$, $b = 2$, $k = 3$, showed that $(a-b)^{(kp)} = a^{(kp)} - kab + b^{(kp)}$ was **not** an identity:

$$(4-2)^{(3p)} = 2^{(3p)} = 5 \text{ while } 4^{(3p)} - 3\cdot 4\cdot 2 + 2^{(3p)} = 22 - 24 + 5 = 3$$

So much for jumping the gun! Believing now that if there was a single identity that involved the terms $(a-b)^{(kp)}$, $a^{(kp)}$, and $b^{(kp)}$, I would have to take a more organized approach. I decided to look at particular examples in a more orderly fashion than I had with Tables 1, 2 and 3.

a	b	$(a-b)^{(1p)}$	$a^{(1p)}$	$b^{(1p)}$	$a^{(1p)} - ab + b^{(1p)}$	$(a-b)^{(1p)} - \left(a^{(1p)} - ab + b^{(1p)}\right)$
5	5	0	15	15	5	−5
5	4	1	15	10	5	−4
5	3	3	15	6	6	−3
5	2	6	15	3	8	−2
5	1	10	15	1	11	−1

Table 4

a	b	$(a-b)^{(3p)}$	$a^{(3p)}$	$b^{(3p)}$	$a^{(3p)} - 3ab + b^{(3p)}$	$(a-b)^{(3p)} - \left(a^{(3p)} - 3ab + b^{(3p)}\right)$
5	5	0	35	35	−5	5
5	4	1	35	22	−3	4
5	3	5	35	12	2	3
5	2	12	35	5	10	2
5	1	22	35	1	21	1

Table 5

a	b	$(a-b)^{(4p)}$	$a^{(4p)}$	$b^{(4p)}$	$a^{(4p)}-4ab+b^{(4p)}$	$(a-b)^{(4p)}-\left(a^{(4p)}-4ab+b^{(4p)}\right)$
5	5	0	45	45	−10	10
5	4	1	45	28	−7	8
5	3	6	45	15	0	6
5	2	15	45	6	11	4
5	1	28	45	1	26	2

Table 6

These results suggested that, for every positive integer k, an identity that would generalize the familiar identity, $(a-b)^2 = a^2 - 2ab + b^2$, for all polygonal numbers would look like:

$$(a-b)^{(kp)} = a^{(kp)} - kab + b^{(kp)} + (k-2)b \qquad (5)$$

A proof that (5) is actually an identity follows:

$$\begin{aligned}(a-b)^{(kp)} &= a - b + k\frac{(a-b)^2-(a-b)}{2}\\ &= a + b - 2b + k\frac{a^2 - a - 2ab + b^2 - b + 2b}{2}\\ &= \left(a + k\frac{a^2-a}{2}\right) - kab + \left(b + k\frac{b^2-b}{2}\right) + (k-2)b\\ &= a^{(kp)} - kab + b^{(kp)} + (k-2)b\end{aligned}$$

Notice again that in the special case $k = 2$, we get

$$(a-b)^{(2p)} = (a-b)^2 = a^2 - 2ab + b^2 = a^{(2p)} - 2ab + b^{(2p)} + (2-2)b$$

And what about $a^2 - b^2 = (a+b)(a-b)$?

In addition to $(a+b)^2 = a^2 + 2ab + b^2$ and $(a-b)^2 = a^2 - 2ab + b^2$, there's one more basic identity with two variables that first-year algebra students must master. Often referred to as the difference of two perfect squares, that identity reads:

$$a^2 - b^2 = (a+b)(a-b) \qquad (6)$$

Once again, I resorted to tables of results with $k = 3$ and 4 to see if I could come up with a similar identity for the partial sums of other arithmetic progressions:

a	b	$a^{(3p)}$	$b^{(3p)}$	$a^{(3p)}-b^{(3p)}$	$(a+b)(a-b)$	$a^{(3p)}-b^{(3p)}-(a+b)(a-b)$
5	5	35	35	0	0	0
5	4	35	22	13	9	4
5	3	35	12	23	16	7
5	2	35	5	30	21	9
5	1	35	1	34	24	10

Table 7

a	b	$a^{(4p)}$	$b^{(4p)}$	$a^{(4p)}-b^{(4p)}$	$(a+b)(a-b)$	$a^{(4p)}-b^{(4p)}-(a+b)(a-b)$
5	5	45	45	0	0	0
5	4	45	28	17	9	8
5	3	45	15	30	16	14
5	2	45	6	39	21	18
5	1	45	1	44	24	20

Table 8

Noting that the second differences between adjacent values in the last columns of Tables 7 and 8 were constants, I was confident that such an identity existed and involved a quadratic relationship. However, just what that pattern was in terms of the variables k, a and b wasn't so clear at first. I decided to focus my efforts on finding a relationship between the values in the second and last columns of Tables 7 and 8.

b	$5^{(3p)} - b^{(3p)} - (5+b)(5-b)$
5	0
4	4
3	7
2	9
1	10

Table 10

b	$5^{(4p)} - b^{(4p)} - (5+b)(5-b)$
5	0
4	8
3	14
2	18
1	20

Table 11

Using standard procedures, such as simultaneous equations or the regression feature of a graphing calculator, for finding a quadratic relationship, one can find that

$$5^{(3p)} - b^{(3p)} - (5+b)(5-b) = \frac{(5-b)(b+4)}{2}$$
$$5^{(4p)} - b^{(4p)} - (5+b)(5-b) = (5-b)(b+4)$$

Equivalently, I was able to write

$$5^{(3p)} - b^{(3p)} = (5+b)(5-b) + \frac{(5-b)(b+4)}{2} \qquad (7)$$
$$5^{(4p)} - b^{(4p)} = (5+b)(5-b) + (5-b)(b+4) \qquad (8)$$

Noting in Tables 7 and 8 that a had the value 5 for all values of b, I first replaced the values 5 and 4 in equations (7) and (8) with the variable expressions a and $a-1$ and then proceeded to simplify the right-hand side of each equation to come up with

$$\begin{aligned} a^{(3p)} - b^{(3p)} &= (a+b)(a-b) + \frac{(a-b)(b+a-1)}{2} \\ &= \frac{3(a+b)(a-b) - (a-b)}{2} \\ &= \frac{(3(a+b)-1)(a-b)}{2} \end{aligned}$$

Similarly, we have

$$\begin{aligned} a^{(4p)} - b^{(4p)} &= (a+b)(a-b) + (a-b)(b+a-1) \\ &= (2(a+b)-1)(a-b) \\ &= \frac{(4(a+b)-2)(a-b)}{2} \end{aligned}$$

With these results in mind, I made the conjecture that, in general, the equation

$$a^{(kp)} - b^{(kp)} = \frac{(k(a+b)+2-k)(a-b)}{2} \qquad (9)$$

is an identity. A proof that this is the case follows. Recall first that we previously showed that

$$n^{(kp)} = 1 + (1+k) + (1+2k) + \cdots + (1+(n-1)k) = n + k\frac{(n^2-n)}{2}$$

Accordingly,

$$
\begin{aligned}
a^{(kp)} - b^{(kp)} &= a + k\frac{(a^2-a)}{2} - \left(b + k\frac{(b^2-b)}{2}\right) \\
&= \frac{2a + ka^2 - ka - 2b - kb^2 + kb}{2} \\
&= \frac{k(a^2-b^2) + 2(a-b) - k(a-b)}{2} \\
&= \frac{(k(a+b) + 2 - k)(a-b)}{2}
\end{aligned}
$$

Table 12 summarizes the extensions of the familiar formulas with quadratic expressions presented in this article.

Standard Formulas	Extended Formulas
$(a+b)^2 = a^2 + 2ab + b^2$	$(a+b)^{(kp)} = a^{(kp)} + kab + b^{(kp)}$
$(a-b)^2 = a^2 - 2ab + b^2$	$(a-b)^{(kp)} = a^{(kp)} - kab + b^{(kp)} + (k-2)b$
$a^2 - b^2 = (a+b)(a-b)$	$a^{(kp)} - b^{(kp)} = \frac{(k(a+b)+2-k)(a-b)}{2}$

Table 12

At this point, I thought I was done with this topic. However, looking back on my work, I noticed that while I had assumed k was a positive integer, that assumption didn't appear to be necessary in my proofs that equations (4), (5) and (9) were identities. This led me to work with some of the values in Table 13 generated by replacing k with the integers from -2 to 3.

n	1	2	3	4	5	6
$n^{(-2p)}$	1	0	-3	-8	-15	-24
$n^{(-1p)}$	1	1	0	-2	-5	-9
$n^{(0p)}$	1	2	3	4	5	6
$n^{(1p)}$	1	3	6	10	15	21
$n^{(2p)}$	1	4	9	16	25	36
$n^{(3p)}$	1	5	12	22	35	50

Table 13

Check $(a+b)^{(kp)} = a^{(kp)} + kab + b^{(kp)}$ for $a = 2$, $b = 3$ and $k = 0$:

$$(2+3)^{(0p)} = 5^{(0p)} = 5 = 2 + 3 = 2^{(0p)} + 0 \cdot 2 \cdot 3 + 3^{(0p)}$$

Check $(a-b)^{(kp)} = a^{(kp)} - kab + b^{(kp)} + (k-2)b$ for $a = 5$, $b = 1$ and $k = -1$:

$$(5-1)^{(-1p)} = 4^{(-1)} = -2 = -5 + 5 + 1 - 3 = 5^{(-1p)} - (-1) \cdot 5 \cdot 1 + 1^{(-1p)} + (-1-2) \cdot 1$$

Check $a^{(kp)} - b^{(kp)} = \frac{(k(a+b)+2-k)(a-b)}{2}$ for $a = 6$, $b = 4$ and $k = -2$:

$$6^{(-2p)} - 4^{(-2p)} = -24 - (-8) = -16 = \frac{(-20+4) \cdot 2}{2} = \frac{\big(-2 \cdot (6+4) + 2 - (-2)\big)(6-4)}{2}$$

It seems that our generalizations of the familiar 2-variable high school identities hold for all integral values of k. Readers are encouraged to share these results with their students and encourage verification for other values of the variables. They are also encouraged to join in the search for extensions and generalizations of other familiar identities.

Highlight on Hooks

Highlight on Hooks is a regular column in the NYSMTJ dedicated to hooks, projects, activities, and best practices in K-16 mathematics classes. Please consider sharing your best "hooks" with our readership – good teachers "borrow" from other teachers. Please send your submissions to Keary Howard, Associate Editor, Department of Mathematical Sciences, SUNY Fredonia, Fredonia, NY 14063 or email them to: keary.howard@fredonia.edu.

Are Lucky Charms® Really Magically Delicious? A Middle School Statistics and Math Project that Feeds the Mind

Holly M. Richardson Keary Howard
SUNY Fredonia

Abstract

This project integrates statistics, data collection, technology, and designing experiments to produce a hands-on and engaging project suitable for middle and high school students alike. Are the "real" Lucky Charms® truly better than generic cereal brands? We empirically explore our hypothesis via marshmallow density and blind taste tests. Teams are required to count the number of marshmallows in a serving of cereal and rate the taste of the cereal in order to determine which cereal is the best. A reproducible laboratory is included.

Introduction

Many students share a displeasure of both mathematics and statistics. Terms and vocabulary such as mean, median, or standard deviation prove difficult without direct and relevant application. It is important to ease their nerves regarding statistics, and engaging labs can be the perfect prescription. In addition, labs and projects are a great opportunity for the teacher to be creative while meeting curricular requirements. Once the data are collected, students can use their calculators as a tool to produce graphs and analyze what they have created. Students will also explore the scientific method by creating a final lab write up. Students will develop a research question, state the hypothesis, explain the experimental design, and state their results and conclusion. In addition, they are required to defend their conclusion using the data that they gathered in previous components of the laboratory.

Magically delicious? How would you know?

This project can be broken into three tasks: Collecting and documenting data regarding the number of marshmallows in different brands of cereal, collecting and documenting data regarding the taste of each cereal brand, and defending their findings in a final lab write up. Each task is summarized below:

- *How magically delicious are Lucky Charms® ?*

In this portion of the lab, students are required to count the number of marshmallows in a cup of Lucky Charms® and alternative brands of similar cereal. Students will record the number of whole marshmallows their serving contained before compiling their data in a class set of data, an example of which is shown in Table 1.

Lucky Charms®	Marshmallow Glitters®	Marshmallow Matey's®
21	20	10
22	25	13
17	18	20
15	27	17

24	29	12
16	20	11
18	18	12
11	17	9
19	16	10

Table 1

From the table, the students could calculate and interpret the data, which could be validated with a calculator later. They could be asked a variety of questions, some of which are listed below:

I. Which brand has the highest marshmallow density based on the mean? What is the mean value?

II. Create a box and whisker plot for each set of data on the same axes so that they can be easily compared.

III. Interpret your results and decide which cereal would be a better choice and why.

- *Taste test.*

This second component of the lab requires students to taste each brand of cereal and rate them on a lichert scale from 1 to 5. Once the students have determined the rating, a classwide set of data will be displayed in a table. An example is shown in Table 2.

Lucky Charms®	Marshmallow Glitters®	Marshmallow Matey's®
4	2	2
5	2	3
4	1	2
3	2	1
4	1	1
5	3	3
4	1	1
5	1	3
5	2	2

Table 2

Then, the students will create box and whisker plots to analyze the data and asnwer questions regarding the data. Sample questions are listed below:

I. Which brand has the lowest rating based on the mean? What is the mean value?

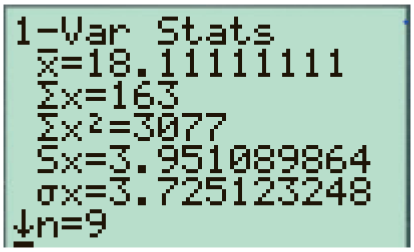
II. Which brand has the highest variability? How do you know?

- *Lab write up.*

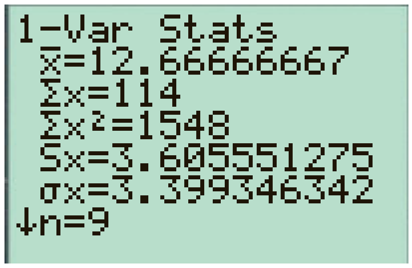
In the final portion on the lab, students will explore the scientific method. Students will develop a research question, state their hypotheses, describe the experiment they performed and how they collected data, display their results, and defend their conclusion using evidence.

Magical TI's

Now that the students have a set of data and have made calculations, they can take advantage of the TI-84. The students can enter the data into the calculator and check their previous work. The students can receive the basic descriptive statistics for both the Lucky Charms® and Marshmallow Matey's® data, which is shown below:

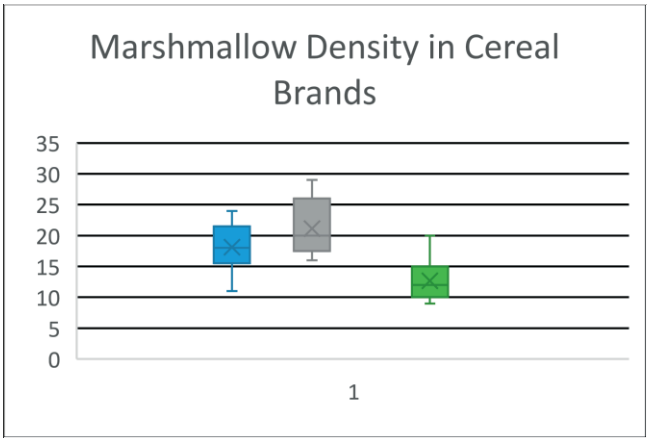


Basic descriptive statistics for Lucky Charms®



Basic descriptive statistics for Marshmallow Matey's®

The students can also check their handmade box and whisker plots by creating them on the calculator. Below are the box and whisker plots for the three brands of cereal and the number of marshmallows they contained.

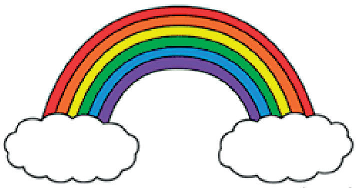


Box and whisker plots for Lucky Charms® (left), Marshmallow Glitters® (middle) and Marshmallow Matey's® (right)

Once the box and whisker plots are corrected if need be, the students can make informed decisions as to whether or not Lucky Charms® are magically delicious.

Conclusion

Hands-on activities are critical in developing conceptual mathematical knowledge, particularly with key universal topics such as descriptive statistics. Not only are labs and projects more fun, but they make the material less intimidating to the students. For middle school students, aspects of this lab may want to be done as a class to ensure the students are not only staying on task, but they are fully understanding the new terms and concepts. At the high school level, students would be given more freedom and could work in small groups to complete the lab. As an additional project, students could be required to gather their own data and present their findings to the class. In our iteration of the lab, the results were mixed. Marshmallow Glitters® produced a higher marshmallow density, but failed to pass the ultimate blind taste test, where evidence favored the 'real' Lucky Charms®. What conclusions will your students draw? Have fun and enjoy a tasty treat as you deepen your knowledge of descriptive statistics and experimental design.



Part One: How Magically Delicious are Lucky Charms®?

The first way to determine if Lucky Charms® are actually magically delicious is marshmallow density. The more marshmallows, the more magically delicious they are!

You will be given three bags, each containing one brand of cereal. Your first task is to count the number of WHOLE marshmallows each bag contains. Fill in your data in the table below:

Brand of Cereal	A	B	C
Marshmallow Density			

Now, copy down the class data below:

	Brand A	Brand B	Brand C
Group 1			
Group 2			
Group 3			
Group 4			
Group 5			
Average for Each Brand			

Create a box and whisker plot for each brand of cereal. Be sure to include labels! The table below will help you determine the values needed for the box and whisker plots.

	Brand A	Brand B	Brand C
Minimum			
First Quartile			
Mean			
Third Quartile			
Maximum			

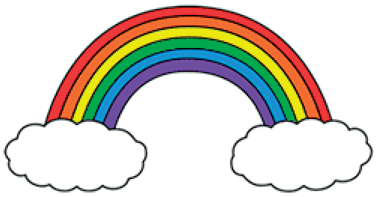
Compare the data using the box and whisker plots to answer the following questions.

1. Which brand has the highest marshmallow density based on the mean? What is the mean value?

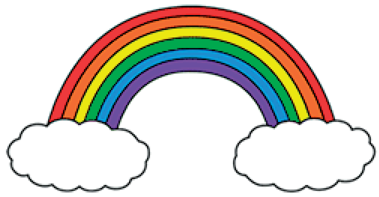
2. Which brand has the lowest marshmallow density based on the mean? What is the mean value?

3. Which brand has the most consistent value of marshmallow density? How do you know?

4. Based on the box and whisker plots, which cereal would you prefer? Why?



Part Two: Taste Test



It's important that the cereal should contain many marshmallows … but it should also taste great!

You will sample each brand of cereal, but you will not know which brand it is until the end. Taste at least 2 oat pieces and 2 marshmallow pieces from each brand, then give it an overall rating between 1 and 5. A rating of 1 means it was awful/disgusting, and 5 means it was delicious! Complete the chart below:

Brand	A	B	C
Rating			

Now, copy down the class data below:

	Brand A	Brand B	Brand C
Group 1			
Group 2			
Group 3			
Group 4			
Group 5			
Average for Each Brand			

Create a box and whisker plot for each brand of cereal. Be sure to include labels! The table below will help you determine the values needed for the box and whisker plots.

	Brand A	Brand B	Brand C
Minimum			
First Quartile			
Mean			
Third Quartile			
Maximum			

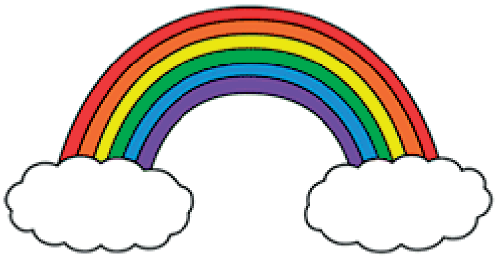

Compare the data using the box and whisker plots to answer the following questions.

1. Which brand has the highest rating based on the mean? What is the mean value?

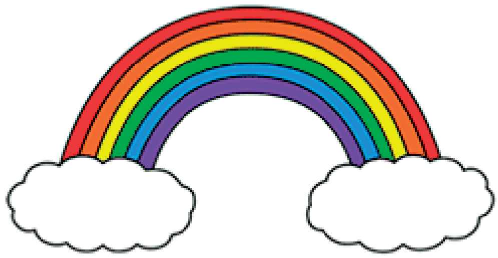
2. Which brand has the lowest rating based on the mean? What is the mean value?

3. Which brand has the highest variability? How do you know?

4. Based on the box and whisker plots, which brand of cereal would you prefer? Why?



Part Three: Lab Write Up



Now that we have collected some data, let's use it!

1. What is the research question that you want to explore?

2. In the space below, write your hypothesis. (Do you think Lucky Charms will be magically delicious? Write a sentence to describe what you think the results will be.)

3. How will you determine if your hypothesis is correct? Describe the experiment that we did. (Someone should be able to repeat the experiment based on your description. Be specific!)

4. Describe the results of the experiment below.

5. Was your hypothesis correct? What cereal would you recommend to someone who was looking for the most magically delicious cereal?

Problems and Solutions

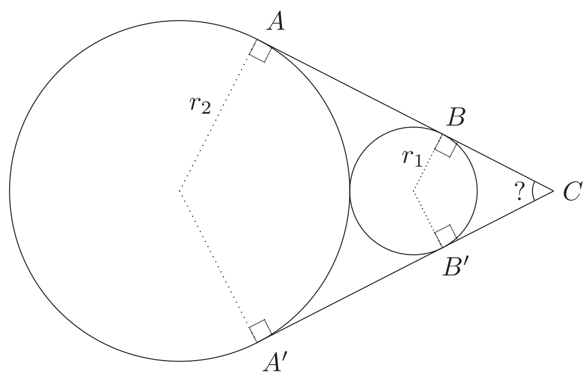
Harris Kwong
Problems Editor

Problems appropriate for this column are assumed not to be widely known nor accessible in popular textbooks. Those problems that are interesting or useful to teachers of mathematics by virtue of illustrating key mathematical concepts are especially welcome. Solutions to the problems proposed in this issue should be submitted on separate signed sheets no later than October 15, 2017. All correspondence should be sent to Prof. Harris Kwong, Department of Mathematical Sciences, SUNY Fredonia, Fredonia, NY, 14063 (Harris.Kwong@fredonia.edu).

PROBLEMS

534. Proposed by Robert Serkey, Westinghouse High School (retired), Brooklyn, NY.

Two externally tangent circles have radii r_1 and r_2 with $r_2 > r_1$. Express, in terms of r_1 and r_2, the sine of the angle formed by the common external tangents. Give the answer in simplest form.

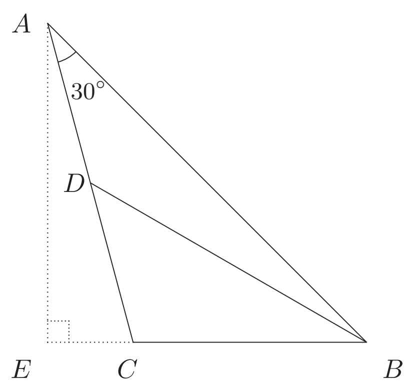


535. Proposed by David Hankin, New York Math Circle, New York, NY.

Define an *ascending number* to be a positive integer with two or more digits, each of whose digits except the last is less than the digit to its right. Find the number of ascending integers.

536. Proposed by James M. Parks, SUNY Potsdam (retired), Potsdam, NY.

Given $\triangle ABC$ with $\angle B$ acute, $m\angle A = 30°$, and median BD to side AC from B, equal to altitude AE from A to BC, determine $m\angle B$.



537. Proposed by Serkey.

If $y > 0$, and

$$(49y)^{\log 7} - (25y)^{\log 5} = 0,$$

solve for y in the form of $y = \frac{1}{p}$, where p is an integer.

(NOT SO) ELEMENTARY

Problems proposed for this category should require ingenuity but not make use of advanced theorems. Proposals are solicited from any source, but **solutions will be accepted only from primary school teachers or precollege students**.

NSE 84. Proposed by the Section Editor.

A farmer sells two horses for \$990 each. He gains 10% on one sale, but loses 10% on the other sale. What is his profit or loss on the whole transaction?

SOLUTIONS

522. Proposed by David Hankin, New York Math Circle, New York, NY.

The polynomial $P(x)$ has integer coefficients. Prove that if $P(x) = 10$ for four distinct integer values of x, then $P(x)$ cannot equal 27 for any integer value of x.

Solution by David E. Manes, SUNY Oneonta (retired), Oneonta, NY.

Assume that the four distinct integer values of x such that $P(x) = 10$ are a, b, c, ad d. Then the polynomial $P(x) - 10$ has four integer zeroes a, b, c, and d. Hence,

$$P(x) - 10 = (x-a)(x-b)(x-c)(x-d) \cdot g(x)$$

for some polynomial $g(x)$ with integer coefficients. Assume to the contrary that $P(z) = 27$ for some integer z. Then

$$17 = P(z) - 10 = (z-a)(z-b)(z-c)(z-d) \cdot g(z),$$

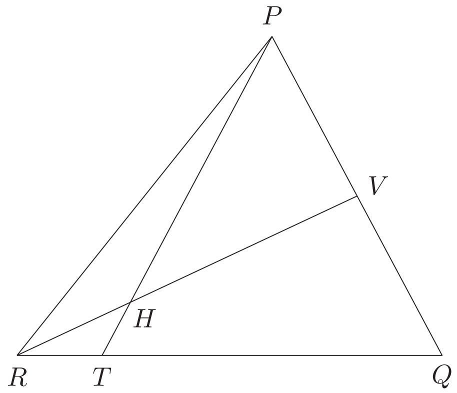
where $z-a$, $z-b$, $z-c$, and $z-d$ are four distinct factors of 17. Since 17 is a prime, it follows that at least three of the four integers $z-a$, $z-b$, $z-c$, and $z-d$ are equal to 1 or -1, in which case two of them would be equal, a contradiction. Hence, $P(x) \neq 27$ for any integer value of x.

Editor's Note. The proposer remarked that this was a modification of a problem that he read in The USSR Olympiad Problem Book. Doug Cashing noticed that the problem can be generalized by replacing the numbers 10 and 27 with any pair of integers whose difference is a prime (or the negative of a prime).

Also solved by Doug Cashing, St. Bonaventure University, St. Bonaventure, NY; Judith Khan, James Madison High School (retired), Brooklyn, NY; Kathleen E. Lewis, University of the Gambia, Republic of the Gambia; Robert Serkey, Westinghouse High School (retired), Brooklyn, NY; Ray Siegrist, SUNY Oneonta, Oneonta, NY; Dave Van Leeuwen, Chatham High School (retired), Chatham, NY; and the proposer.

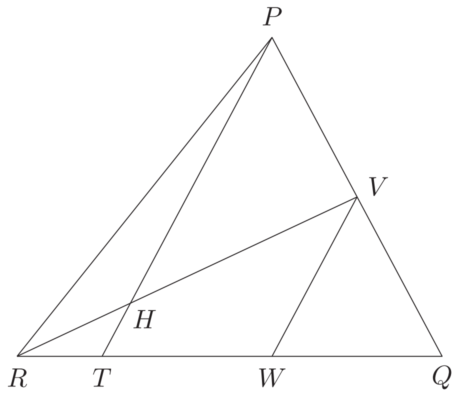
523. Proposed by Robert Serkey, Westinghouse High School (retired), Brooklyn, NY.

In an acute triangle PQR, the point T lies on RQ such that $TQ : RT = 4 : 1$. Let V be the midpoint of PQ, and assume RV intersects PT at H. Find the ratio of the area of the triangle THR to the area of the quadrilateral $HVQT$.



Solution 1 by Kathleen E. Lewis, University of the Gambia, Republic of the Gambia.

Let A be the area of triangle PQR and let x be the length of segment RT. Then RQ has length $5x$. Draw a line segment parallel to PT from V to the point W on the segment RQ.



Triangle TPQ has the same height as RPQ, but its base is $4x$ instead of $5x$, so its area is $\frac{4}{5}A$. Triangles WVQ and TPQ are similar with sides in ratio $1 : 2$, and therefore areas in ratio $1 : 4$, so the area of WVQ is

$$\frac{1}{4} \cdot \frac{4}{5}A = \frac{1}{5}A.$$

The two triangles RVQ and WVQ have the same height, and so do the triangles RPQ and TPQ; hence the ratio of the two heights is $1 : 2$. Since triangle RVQ has the same base as RPQ, but half the height, its area is $\frac{1}{2}A$. Then the area of RVW is

$$\frac{1}{2}A - \frac{1}{5}A = \frac{3}{10}A.$$

Triangles RHT and RVW are similar, with sides in the ratio $1 : 3$, so the area of RHT is

$$\frac{1}{9} \cdot \frac{3}{10}A = \frac{1}{30}A.$$

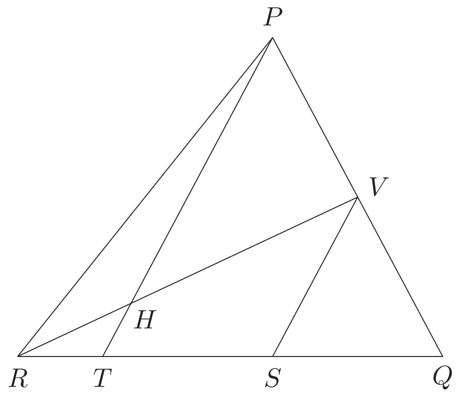
Thus the quadrilateral $HVQT$ has area

$$\frac{1}{2}A - \frac{1}{30}A = \frac{14}{30}A,$$

so the ratio we want is $1 : 14$.

Solution 2 by Dave Van Leeuwen, Chatham High School (retired), Chatham, NY.

We began by adding two pieces to the diagram. These are the midpoint S of the segment QT, and the segment connecting S to V. Originally, we have $RT : TQ = 1 : 4$. We can now extend that to the ratio $RT : TS : SQ = 1 : 2 : 2$.



Since S and V are midpoints, respectively, of the two sides of $\triangle PQT$, the segment connecting them must be parallel to the third side of that triangle, the segment PT. This unlocks the problem. We now have segment TH parallel to the side VS in $\triangle RVS$, so it must divide the two sides RS and RV proportionally. Since $RT : TS = 1 : 2$, we know that RH is one-third the length of RV. Working with the areas of the triangles we can verify the following:

$$\begin{aligned}
\text{area of } \triangle HRT &= \tfrac{1}{2} \cdot RH \cdot RT \cdot \sin(\angle HRT) \\
&= \tfrac{1}{2} \cdot \tfrac{1}{3}\, RV \cdot \tfrac{1}{5}\, RQ \cdot \sin(\angle HRT) \\
&= \tfrac{1}{15} \cdot \tfrac{1}{2} \cdot RV \cdot RQ \cdot \sin(\angle HRT) \\
&= \tfrac{1}{15}\,(\text{area of } \triangle VRQ).
\end{aligned}$$

Since the ratio of the areas of $\triangle RHT$ to $\triangle VRQ$ is $1 : 15$, we can subtract the area of the smaller triangle from the larger, and leave us with the ratio of the areas of $\triangle RHT$ and quadrilateral $HVQT$ as $1 : 14$.

Editor's Note. Dave Van Leeuwen observed that the proof also works when the angle Q is obtuse.

Also solved by Joel Fazekas, Richmond Hill High School (retired), Queens, NY; David Hankin, New York Math Circle, New York, NY; Judith Khan, James Madison High School (retired), Brooklyn, NY; David E. Manes, SUNY Oneonta (retired), Oneonta, NY; Ray Siegrist, SUNY Oneonta, Oneonta, NY; Alex R. Stepanski, Westmoreland High School, Westmoreland, NY; and the proposer.

524. Proposed by David E. Manes, SUNY Oneonta (retired), Oneonta, NY.

Find all the integer solutions of the system of equations

$$\begin{aligned}
x + y + z &= 3, \\
x^3 + y^3 + z^3 &= 3.
\end{aligned}$$

Solution by David Hankin, New York Math Circle, New York, NY.

Use the factoring

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - xz).$$

Then $3 - 3xyz = 3(x^2 + y^2 + z^2 - xy - yz - xz)$, so

$$\begin{aligned}
1 - xyz &= x^2 + y^2 + z^2 - xy - yz - xz \\
&= (x + y + z)^2 - 3xy - 3yz - 3xz \\
&= 9 - 3xy - 3yz - 3xz.
\end{aligned}$$

Therefore,

$$\begin{aligned}
-8 &= xyz - 3xy - 3yz - 3xz \\
&= xyz - 3xy - 3yz - 3xz + 9x + 9y + 9z - 27 \\
&= (x - 3)(y - 3)(z - 3).
\end{aligned}$$

Let $a = x - 3$, $b = y - 3$, and $c = z - 3$. Then $abc = -8$ and $a + b + c = x + y + z - 9 = -6$. The first of these equations implies that $(|a|, |b|, |c|)$ must be a permutation of $(1, 1, 8)$, $(1, 2, 4)$, or $(2, 2, 2)$.

- When $\{|a|, |b|, |c|\} = \{1, 1, 8\}$, then $\{a, b, c\} = \{-1, 1, 8\}$, $\{1, 1, -8\}$, or $\{-1, -1, -8\}$.
- When $\{|a|, |b|, |c|\} = \{1, 2, 4\}$, then $\{a, b, c\} = \{-1, 2, 4\}, \{1, -2, 4\}, \{1, 2, -4\}$, or $\{-1, -2, -4\}$.
- When $\{|a|, |b|, |c|\} = \{2, 2, 2\}$, then $\{a, b, c\} = \{-2, 2, 2\}$ or $\{-2, -2, -2\}$.

Of these sets, the ones whose element-sum is -6 are $\{1, 1, -8\}$ and $\{-2, -2, -2\}$. Thus (a, b, c) must be any permutation of $(1, 1, -8)$ or $(-2, -2, -2)$, and so (x, y, z) is any permutation of $\{4, 4, -5\}$ or $\{1, 1, 1\}$. The requested solutions are therefore $(4, 4, -5)$, $(4, -5, 4)$, $(-5, 4, 4)$, and $(1, 1, 1)$.

Editor's Remark. As suggested by Judith Khan, the factorization $-8 = (x - 3)(y - 3)(z - 3)$ can be obtained directly as follows. From expanding $3^3 = [x + (y + z)]^3$, we find

$$\begin{aligned} 27 &= x^3 + 3x^2(y + z) + 3x(y + z)^2 + (y + z)^3 \\ &= x^3 + 3x^2(y + z) + 3x(y + z)^2 + y^3 + 3y^2z + 3yz^2 + z^3. \end{aligned}$$

But $x^3 + y^3 + z^3 = 3$. Thus,

$$\begin{aligned} 24 &= 3x^2(y + z) + 3x(y + z)^2 + 3yz(y + z) \\ &= 3(y + z)[x^2 + (y + z)x + yz] \\ &= 3(y + z)(x + y)(x + z) \\ &= 3(3 - x)(3 - z)(3 - y). \end{aligned}$$

Also solved by Lawrence Cohen, Nassau Community College (retired), Garden City, NY; Bernard G. Hoerbelt, Genesee Community College (retired), Batavia, NY; Judith Khan, James Madison High School (retired), Brooklyn, NY; Robert Serkey, Westinghouse High School (retired), Brooklyn, NY; Ray Siegrist, SUNY Oneonta, Oneonta, NY; Alex R. Stepanski, Westmoreland High School, Westmoreland, NY; and the proposer.

525. Proposed by Robert Serkey, Westinghouse High School (retired), Brooklyn, NY.

Let $p(x) = \dfrac{x - 7}{2x + 3}$, and let $r(x)$ be a rational function such that $r\big(3p(x)\big) = p(4x + 1)$. Find $r(x)$.

Solution 1 by Alex R. Stepanski, Westmoreland High School, Westmoreland, NY.

Since $3p(x) = \frac{3x-21}{2x+3}$, and $p(4x + 1) = \frac{4x-6}{8x+5}$, we have

$$r\left(\frac{3x - 21}{2x + 3}\right) = \frac{4x - 6}{8x + 5}.$$

Let $y = \frac{3x-21}{2x+3}$, and solve for x. We find $x = \frac{3y+21}{3-2y}$. Substitution yields

$$r(y) = \frac{4\left(\frac{3y+21}{3-2y}\right) - 6}{8\left(\frac{3y+21}{3-2y}\right) + 5} = \frac{24y + 66}{14y + 183}.$$

Hence, $r(x) = \frac{24x+66}{14x+183}$.

Solution 2 by Lisa Lewis, Roslyn High School (retired), Roslyn, NY.

Let

$$m(x) = p(4x + 1) = \frac{4x - 6}{8x + 5},$$

and

$$q(x) = 3p(x) = \frac{3x - 21}{2x + 3}.$$

By interchanging variables, it can be shown that

$$q^{-1}(x) = \frac{3x+21}{3-2x}.$$

It is given that $(r \circ q)(x) = r(q(x)) = m(x)$. Then $\left(r \circ q \circ q^{-1}\right)(x) = \left(m \circ q^{-1}\right)(x)$. Since $\left(q \circ q^{-1}\right)(x) = x$,

$$r(x) = \left(r \circ q \circ q^{-1}\right)(x) = \left(m \circ q^{-1}\right)(x) = \frac{4\left(\frac{3x+21}{3-2x}\right) - 6}{8\left(\frac{3x+21}{3-2x}\right) + 5} = \frac{24x+66}{14x+183}.$$

Also solved by Joel Fazekas, Richmond Hill High School (retired), Queens, NY; Bernard G. Hoerbelt, Genesee Community College (retired), Batavia, NY; Judith Khan, James Madison High School (retired), Brooklyn, NY; Kathleen E. Lewis, University of the Gambia, Republic of the Gambia; David E. Manes, SUNY Oneonta (retired), Oneonta, NY; Ray Siegrist, SUNY Oneonta, Oneonta, NY; and the proposer. One incorrect solution was also received.

NSE 81. Proposed by Bernard G. Hoerbelt, Genesee Community College (retired), Batavia, NY.

My dog, Fido (a smart dog), attempts to solve a quadratic equation in standard form $ax^2 + bx + c = 0$, but carelessly interchanges the constant term, c, and the coefficient, a, of the x^2 term. Remarkably he still gets one of the zeros correct. The incorrect zero he obtains is 5. Find the zeros of the original quadratic equation.

Solution by Westmoreland High School AP Calculus Class, Westmoreland, NY.

The roots of $ax^2 + bx + c = 0$ are

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad \text{and} \qquad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The roots of $cx^2 + bx + a = 0$ are

$$x_3 = \frac{-b + \sqrt{b^2 - 4ac}}{2c} \qquad \text{and} \qquad x_4 = \frac{-b - \sqrt{b^2 - 4ac}}{2c}.$$

Since

$$x_1 x_4 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \cdot \frac{-b - \sqrt{b^2 - 4ac}}{2c} = \frac{b^2 - (b^2 - 4ac)}{4ac} = \frac{4ac}{4ac} = 1,$$

and

$$x_2 x_3 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \cdot \frac{-b + \sqrt{b^2 - 4ac}}{2c} = \frac{b^2 - (b^2 - 4ac)}{4ac} = \frac{4ac}{4ac} = 1,$$

we see that $x_3 = \frac{1}{x_2}$ and $x_4 = \frac{1}{x_1}$. So when a and c in the equation $ax^2 + bx + c = 0$ are switched, the roots are inverted. So if 5 is a root of $cx^2 + bx + a = 0$, the reciprocal $\frac{1}{5}$ must be a root of the original equation. The other root that Fido got correct must equal to its reciprocal. There are only two such numbers: 1 and -1. This means that the original roots could be $\frac{1}{5}$ and 1, or $\frac{1}{5}$ and -1.

Also solved by the proposer.

The Second New^3 Math Conference

Coming Together for Learning, Teaching and Students

A Joint Conference of the Associations of Mathematics Teachers Of

New York - AMTNYS   New Jersey - AMTNJ   New England - ATMNE

HOSTED BY

SAVE the DATES --July 9 – 12, 2017 at Siena College (near Albany, NY)

Conference Program

- Exciting program featuring outstanding mathematics educators from the northeast
- Narrow Grade Band Workshops.
  At each presentation time there will be a Pre-K – K workshop,
  grade 1-2 workshop,
  grade 3-4 workshop,
  grade 5-6 workshop, and
  grade 7-8 and HS workshops. These will be in-depth multi-part strands.

Sample Comments from 1st New^3

The presenters were incredible.

Fantastic speakers, well organized in grade level bands, great opportunities for networking.

This was my first math conference but I loved it. The presenters were excellent. … I raved about it to my principal and superintendent.

Definitely want to bring a team of teachers back next year in 2017

This conference was FABULOUS. I made connections that I never thought possible and I learned a LOT

The conference was very well organized. The sessions were informative. It was one of the best conferences I have ever attended!

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- Community Atmosphere
- Monday social events include a 1. Lake George dinner cruise and 2. concert at Tanglewood.

The New^3 Math Associations will be posting more info about this conference on their websites and sending e-blasts. If you would like to be on a conference distribution list for the latest updates or have questions, please email conference coordinator, Jim Matthews, at matthews@siena.edu. Please use "New Cubed - 2017" in the subject line of your email.

ASSOCIATION OF MATHEMATICS TEACHERS OF NEW YORK STATE
186 Bittersweet Drive, Farmington, NY 14425

To submit a proposal, visit www.amtnys.org.
Deadline for submission is May 19, 2017.

Questions? Contact

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